Duality between Dyadic H^1 and Dyadic BMO via Bellman Function Techniques

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1 Introduction

In Lecture 10 we proved the main part of the Fefferman–Stein theorem $\big(H^1(\mathbf{R}^d)\big)^* = \mathrm{BMO}(\mathbf{R}^d)$, namely that for $b \in L^1_{\mathrm{loc}}$ the pairing with b is bounded on H^1 if and only if $b \in \mathrm{BMO}$, through atoms and the John–Nirenberg inequality. Lectures 13 to 16 built the dyadic counterparts of these objects, namely the martingale $(\mathbb{E}_k f)_k$ of dyadic averages and its differences $\mathbb{D}_k f$, the dyadic square function $S_{\mathcal{D}} f$ and Hardy space $H^1_{\mathcal{D}}$, dyadic $\mathrm{BMO}_{\mathcal{D}}$, and the Haar system. This report presents the dyadic version of the Fefferman–Stein theorem, $\big(H^1_{\mathcal{D}}(I_0)\big)^* = \mathrm{BMO}_{\mathcal{D}}(I_0)$ on a dyadic interval I_0, following Slavin and Volberg [1]. The main statement is Theorem 2.10. Its hard direction is the estimate $\big|\int_{I_0} f\varphi\big| \le \sqrt{2}\,\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2}\,\|f\|_{H^1_{\mathcal{D}}}$ for all $f \in L^2(I_0)$ and all $\varphi \in \mathrm{BMO}_{\mathcal{D}}(I_0)$ with $\int_{I_0}\varphi = 0$, which extends to all $f \in H^1_{\mathcal{D}}(I_0)$ once the integral is written as a sum of products of Haar coefficients.

The dyadic setting is incredibly structured in a sense that the dyadic intervals form a tree, the Haar functions form an orthonormal basis, and $\int_{I_0} f\varphi$ is a sum of products of Haar coefficients. This structure allows a proof by the Bellman function method. One proves a single inequality for two abstract functions U and V on the tree by induction over the generations, where the induction step is an inequality for one function $\mathbb{B}$, the Bellman function. As [1] point out, no extremal problem is posed, so the name refers only to the induction on scales. The method gives an explicit constant, and the main inequality (13) in the L^2 oscillation norm is obtained by an induction over the generations rather than through atoms and the John–Nirenberg inequality as in Lecture 10. The authors of [1] use the same $\mathbb{B}$ for the circle and for the multi-dimensional case, see Remark 4.4.

Throughout, $d = 1$ and $I_0 = [0,1)$, since the induction needs a top generation, and all functions are real-valued, so that no complex conjugates appear in $\int_{I_0} f\varphi$ or in the Haar coefficients. Section 2 collects the background and states the theorem, Section 3 introduces the Bellman function and the Key Lemma, and Section 4 proves the Key Lemma and the theorem. The converse direction is standard and only sketched.

2 Background

2.1 Dyadic Intervals and Averages

In Lecture 4, and again in Lecture 13, we defined the dyadic cubes in $\mathbf{R}^d$ as

$$\mathcal{D} := \Big\{ \big[2^{-k}m_1, 2^{-k}(m_1+1)\big) \times \cdots \times \big[2^{-k}m_d, 2^{-k}(m_d+1)\big) : \ k \in \mathbf{Z},\ m_1,\dots,m_d \in \mathbf{Z} \Big\}.$$

Then we defined the generations as

$$\mathcal{D}_k := \{Q \in \mathcal{D} : \ \ell(Q) = 2^{-k}\}$$

for each k, so that $\mathbf{R}^d = \bigcup_{Q\in\mathcal{D}_k} Q$ is a disjoint union. In this project we restrict to $d = 1$. We shall prove the local duality $\big(H^1_{\mathcal{D}}(I_0)\big)^* = \mathrm{BMO}_{\mathcal{D}}(I_0)$ on a fixed dyadic interval I_0, since the Bellman argument in Section 4 is an induction over generations of dyadic intervals and therefore needs a starting generation, namely the root. Let $I_0 \in \mathcal{D}$ be fixed.

Definition 2.1. The set of dyadic intervals rooted in I_0 is

$$\mathcal{D}(I_0) := \{I \in \mathcal{D} : I \subseteq I_0\},$$

and for $J \in \mathcal{D}(I_0)$ we write $\mathcal{D}(J) := \{I \in \mathcal{D} : I \subseteq J\} \subseteq \mathcal{D}(I_0)$.

Here I_0 is the root, that is, the unique element containing all the others. Then the k-th generation of $\mathcal{D}(I_0)$ is

$$\mathcal{D}_k(I_0) := \{I \in \mathcal{D}(I_0) : |I| = 2^{-k}|I_0|\}, \qquad k \in \mathbf{N}_0.$$

Note that $\mathcal{D}_0(I_0) = \{I_0\}$, that each $\mathcal{D}_k(I_0)$ has 2^k intervals, and that for every k the union $I_0 = \bigcup_{I \in \mathcal{D}_k(I_0)} I$ is disjoint.

Definition 2.2. For $I \in \mathcal{D}(I_0)$ we write I_- and I_+ for the left and right halves of I, respectively, and call them the *dyadic children* of I.

If $I \in \mathcal{D}_k(I_0)$ then $I_-, I_+ \in \mathcal{D}_{k+1}(I_0)$, $I = I_- \cup I_+$ disjointly, and $|I_\pm| = \frac{1}{2}|I|$. Let $I \in \mathcal{D}_k(I_0)$ with $k \geq 1$. The generation $\mathcal{D}_{k-1}(I_0)$ partitions I_0, so exactly one $\widehat{I} \in \mathcal{D}_{k-1}(I_0)$ intersects I, and since dyadic intervals are either disjoint or nested (Lecture 4, Exercise) and $|\widehat{I}| > |I|$, it follows that $I \subsetneq \widehat{I}$. We call $\widehat{I}$ the *parent* of I. Since $\widehat{I}_-, \widehat{I}_+ \in \mathcal{D}_k(I_0)$ and $\mathcal{D}_k(I_0)$ is a partition, I is one of $\widehat{I}_-$, $\widehat{I}_+$.

Let $x \in I_0$. For each $k \geq 0$, since $\mathcal{D}_k(I_0)$ partitions I_0, there is a unique $I_k(x) \in \mathcal{D}_k(I_0)$ containing x, and $I_0(x) = I_0$. The intervals $I_k(x)$ all contain x, so any two of them intersect and are therefore nested:

$$I_0 = I_0(x) \supsetneq I_1(x) \supsetneq I_2(x) \supsetneq \cdots, \qquad \{x\} = \bigcap_{k \geq 0} I_k(x). \tag{1}$$

This is precisely the chain of cubes $Q_k(x)$ from Lecture 14 (written $Q_{k,x}$ in Lecture 13), restricted to $k \geq 0$. The root also fixes the exact number of ancestors. Suppose $I \supsetneq J$ with $I \in \mathcal{D}(I_0)$ and $J \in \mathcal{D}_k(I_0)$. Then for any $x \in J$ we have $I = I_i(x)$ for some $i \in \{0, 1, \ldots, k-1\}$. So J has exactly k strict ancestors, one in each generation $0, \ldots, k-1$. In particular a sum over the strict ancestors of J is a finite sum with k terms.

We recall the notation for averages from Lecture 9.

Definition 2.3. Let $f \in L^1(I_0)$ and $I \in \mathcal{D}(I_0)$. Then

$$\langle f \rangle_I := \frac{1}{|I|} \int_I f(x)\,\mathrm{d}x.$$

Since $I = I_- \cup I_+$ disjointly and $|I_\pm| = \frac{1}{2}|I|$, we obtain

$$\begin{aligned} \langle f \rangle_I &= \frac{1}{|I|}\Big(\int_{I_-} f(x)\,\mathrm{d}x + \int_{I_+} f(x)\,\mathrm{d}x \Big) \\ &= \frac{1}{|I|}\big(|I_-|\langle f \rangle_{I_-} + |I_+|\langle f \rangle_{I_+} \big) = \tfrac{1}{2}\big(\langle f \rangle_{I_-} + \langle f \rangle_{I_+} \big). \end{aligned} \tag{2}$$

Identity (2) is the case $d = 1$ of $\langle f \rangle_R = 2^{-d} \sum_{Q \in \mathcal{D}_k,\, Q \subsetneq R} \langle f \rangle_Q$ for $R \in \mathcal{D}_{k-1}$ from Lecture 14, that is, the martingale property of dyadic averages (Lecture 13).

From now on we take $I_0 = [0, 1)$ without loss of generality, by the translation and scaling invariance of Lebesgue measure, see Remark 2.11.

2.2 Haar Functions and Martingale Differences

Recall from Lecture 13 the conditional expectation of $f \in L^1(I_0)$ with respect to $\sigma(\mathcal{D}_k(I_0))$ and, following the convention of Lecture 14, the martingale differences,

$$\mathbb{E}_k f := \sum_{I \in \mathcal{D}_k(I_0)} \langle f \rangle_I \, \chi_I, \qquad \mathbb{D}_k f := \mathbb{E}_k f - \mathbb{E}_{k-1} f, \qquad k \geq 1. \tag{3}$$

Thus $\mathbb{E}_0 f = \langle f \rangle_{I_0}$ is constant on I_0, and for $I \in \mathcal{D}_{k-1}(I_0)$ the function $\chi_I \, \mathbb{D}_k f$ is constant on each of I_-, I_+ and has mean zero on I (Lecture 14, equations (1) and (2) there), hence a multiple of the Haar function h_I defined below. We now express $\mathbb{D}_k f$ in terms of the Haar system of Lecture 16.

Definition 2.4 (Haar functions, Lecture 16)**.** For $I \in \mathcal{D}(I_0)$ the *Haar function* of I is

$$h_I := |I|^{-1/2} \big(\chi_{I_-} - \chi_{I_+} \big),$$

and the family $\{h_I\}_{I \in \mathcal{D}(I_0)}$ is the *Haar system* on I_0.

Definition 2.5 (Haar coefficients)**.** For $f \in L^1(I_0)$ and $I \in \mathcal{D}(I_0)$ the *Haar coefficient* of f at I is the $L^2(I_0)$ inner product

$$\langle f, h_I \rangle := \int_{I_0} f(x)\, h_I(x)\, \mathrm{d}x = \frac{\sqrt{|I|}}{2} \big(\langle f \rangle_{I_-} - \langle f \rangle_{I_+} \big),$$

where the second equality follows from $\int_{I_\pm} f(x)\,\mathrm{d}x = |I_\pm| \langle f \rangle_{I_\pm}$ and $|I_\pm| = \tfrac{1}{2}|I|$.

The following proposition collects what we need about the Haar system. Parts (i) and (ii) are the Proposition and the Remark of Lecture 16, and (iii) follows from them and the dyadic Plancherel theorem (Lecture 14, Theorem 1).

Proposition 2.6.

(i) For $f \in L^1(I_0)$, $k \geq 1$ and $I \in \mathcal{D}_{k-1}(I_0)$,

$$\chi_I \, \mathbb{D}_k f = \langle f, h_I \rangle \; h_I, \qquad \text{hence} \qquad \mathbb{D}_k f = \sum_{I \in \mathcal{D}_{k-1}(I_0)} \langle f, h_I \rangle \; h_I.$$

(ii) The Haar system $\{h_I\}_{I \in \mathcal{D}(I_0)}$ is orthonormal in $L^2(I_0)$.

(iii) For $f \in L^2(I_0)$, in $L^2(I_0)$,

$$f - \langle f \rangle_{I_0} = \sum_{I \in \mathcal{D}(I_0)} \langle f, h_I \rangle \; h_I \qquad \text{and} \qquad \int_{I_0} \big| f(x) - \langle f \rangle_{I_0} \big|^2 \, \mathrm{d}x = \sum_{I \in \mathcal{D}(I_0)} \langle f, h_I \rangle^2 .$$

The same holds with I_0 replaced by any $J \in \mathcal{D}(I_0)$ and $\mathcal{D}(I_0)$ by $\mathcal{D}(J)$.

Proof. Only (iii) remains. Extend f by zero to $\mathbf{R}$. Since $I_0 = [0,1)$ is a union of intervals of $\mathcal{D}_k$ for every $k \geq 0$, the operators $\mathbb{E}_k$ of Lecture 14 restricted to I_0 agree with (3). By Lecture 14, Theorem 1(a), $\mathbb{E}_k f \to f$ in L^2 as $k \to \infty$, so by telescoping

$$f - \langle f \rangle_{I_0} = \lim_{k \to \infty} \big(\mathbb{E}_k f - \mathbb{E}_0 f \big) = \sum_{k \in \mathbf{N}} \mathbb{D}_k f \qquad \text{in } L^2(I_0),$$

and inserting (i) gives the expansion. The $\mathbb{D}_k f$ are mutually orthogonal (Lecture 14, proof of Theorem 1(b)), so

$$\int_{I_0} \big| f(x) - \langle f \rangle_{I_0} \big|^2 \, \mathrm{d}x = \sum_{k \in \mathbf{N}} \int_{I_0} |\mathbb{D}_k f(x)|^2 \, \mathrm{d}x = \sum_{k \in \mathbf{N}} \sum_{I \in \mathcal{D}_{k-1}(I_0)} \langle f, h_I \rangle^2 ,$$

where the last step uses (i) together with (ii), since the h_I of one generation have disjoint supports. For $J \in \mathcal{D}(I_0)$ the same argument applies with J as the root, since $\mathcal{D}(J)$ is again a rooted dyadic lattice. ∎

2.3 The Dyadic Square Function

Recall from Lecture 15 the dyadic square function and from Lecture 16 its truncations, here restricted to I_0 and to generations $k \geq 1$,

$$S_{\mathcal{D}} f := \Big(\sum_{k \in \mathbf{N}} |\mathbb{D}_k f|^2 \Big)^{1/2}, \qquad S_{\mathcal{D},N} f := \Big(\sum_{k=1}^{N} |\mathbb{D}_k f|^2 \Big)^{1/2}, \qquad N \geq 0, \tag{4}$$

where $S_{\mathcal{D},0} f = 0$ is the empty sum, so that $S_{\mathcal{D},N} f$ increases to $S_{\mathcal{D}} f$ pointwise as $N \to \infty$. Let $x \in J \in \mathcal{D}_N(I_0)$. By (1), the intervals $I \in \mathcal{D}_{k-1}(I_0)$ containing x for $k = 1, \dots, N$ are exactly the strict ancestors of J, and $|h_I(x)|^2 = |I|^{-1}$ on I, so Proposition 2.6(i) gives

$$S_{\mathcal{D},N} f(x)^2 = \sum_{\substack{I \in \mathcal{D}(I_0) \\ I \supsetneq J}} \frac{\langle f, h_I \rangle^2}{|I|} = \frac{1}{4} \sum_{\substack{I \in \mathcal{D}(I_0) \\ I \supsetneq J}} \big(\langle f \rangle_{I_-} - \langle f \rangle_{I_+} \big)^2, \qquad x \in J \in \mathcal{D}_N(I_0), \tag{5}$$

which is a finite sum with N terms, constant on each $J \in \mathcal{D}_N(I_0)$. From now on we abbreviate sums over the strict ancestors of J as $\sum_{I \supsetneq J}$. Letting $N \to \infty$ gives the Haar form of $S_{\mathcal{D}} f$ from Lecture 16, and integrating,

$$\int_{I_0} S_{\mathcal{D}} f(x)\, \mathrm{d}x = \frac{1}{2} \int_{I_0} \Big(\sum_{I \ni x} \big(\langle f \rangle_{I_-} - \langle f \rangle_{I_+} \big)^2 \Big)^{1/2} \mathrm{d}x. \tag{6}$$

By the theorem of Lecture 16, applied to $f - \langle f \rangle_{I_0}$ extended by zero to $\mathbf{R}$, whose martingale differences of index $k \leq 0$ vanish, for $1 < p < \infty$ there are constants $A_p, B_p > 0$ such that for all $f \in L^p(I_0)$

$$A_p \big\| f - \langle f \rangle_{I_0} \big\|_{L^p(I_0)} \leq \big\| S_{\mathcal{D}} f \big\|_{L^p(I_0)} \leq B_p \big\| f - \langle f \rangle_{I_0} \big\|_{L^p(I_0)}.$$

The mean is subtracted because $S_{\mathcal{D}}$ in (4) only sums over $k \geq 1$ and therefore vanishes on constants. At the endpoint $p = 1$ the condition $S_{\mathcal{D}} f \in L^1$ is strictly stronger than $f \in L^1$, and it is there that the square function defines a new space, the dyadic Hardy space of Section 2.4.

2.4 Dyadic BMO and Dyadic H^1

Definition 2.7 (Dyadic BMO, Lecture 13)**.** For $\varphi \in L^1(I_0)$ let

$$\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}}} := \sup_{J \in \mathcal{D}(I_0)} \frac{1}{|J|} \int_J \big| \varphi(x) - \langle \varphi \rangle_J \big|\, \mathrm{d}x, \tag{7}$$

and let $\mathrm{BMO}_{\mathcal{D}}(I_0)$ be the space of φ with $\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}}} < \infty$, modulo constants.

By the John–Nirenberg inequality, which holds for $\mathrm{BMO}_{\mathcal{D}}$ with the same proof (Lecture 13), the L^2 oscillation gives an equivalent norm (Lecture 10, Corollary, with $p = 2$),

$$\begin{gathered} \|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2} := \sup_{J \in \mathcal{D}(I_0)} \Big(\frac{1}{|J|} \int_J \big| \varphi(x) - \langle \varphi \rangle_J \big|^2\, \mathrm{d}x \Big)^{1/2}, \\ \|\varphi\|_{\mathrm{BMO}_{\mathcal{D}}} \leq \|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2} \leq C_2 \|\varphi\|_{\mathrm{BMO}_{\mathcal{D}}}. \end{gathered} \tag{8}$$

We shall state the main inequality in the norm $\|\cdot\|_{\mathrm{BMO}_{\mathcal{D}},2}$. Since $I_0 \in \mathcal{D}(I_0)$, (8) gives $\varphi \in L^2(I_0)$, so Proposition 2.6(iii) applies on every $J \in \mathcal{D}(I_0)$ and yields the Haar form

$$\|\varphi\|^2_{\mathrm{BMO}_{\mathcal{D}},2} = \sup_{J \in \mathcal{D}(I_0)} \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \langle \varphi, h_I \rangle^2 = \sup_{J \in \mathcal{D}(I_0)} \frac{1}{4|J|} \sum_{I \in \mathcal{D}(J)} |I| \big(\langle \varphi \rangle_{I_-} - \langle \varphi \rangle_{I_+} \big)^2. \tag{9}$$

Lecture 15 defines $H^1_{\mathcal{D}}(\mathbf{R}^d)$ as the space of $f \in L^1$ with $M_{\mathcal{D}} f \in L^1$, normed by $\|M_{\mathcal{D}} f\|_{L^1}$, and states without proof (Lecture 15, Theorem 2) that this is the same space as $\{f \in L^1 : S_{\mathcal{D}} f \in L^1\}$. We take the square function as the definition.

Definition 2.8 (Dyadic Hardy space)**.**

$$H^1_{\mathcal{D}}(I_0) := \big\{ f \in L^1(I_0) : S_{\mathcal{D}} f \in L^1(I_0) \big\}, \qquad \|f\|_{H^1_{\mathcal{D}}} := \int_{I_0} S_{\mathcal{D}} f(x)\,\mathrm{d}x. \tag{10}$$

Remark 2.9. The quantity $\|f\|_{H^1_{\mathcal{D}}} = \int_{I_0} S_{\mathcal{D}} f(x)\,\mathrm{d}x$ is a seminorm on $H^1_{\mathcal{D}}(I_0)$ which vanishes exactly on the constants: $\mathbb{D}_k c = 0$ for all k, and conversely if $S_{\mathcal{D}} f = 0$ then $\mathbb{D}_k f = 0$ for all k, so $\mathbb{E}_k f = \langle f \rangle_{I_0}$ for all k and $f = \langle f \rangle_{I_0}$ a.e. by Lebesgue differentiation along dyadic intervals (Lecture 15). Hence $H^1_{\mathcal{D}}(I_0)$ is a normed space modulo constants, in the same way as $\mathrm{BMO}_{\mathcal{D}}(I_0)$, and every bounded linear functional on $H^1_{\mathcal{D}}(I_0)$ vanishes on constants. From now on $H^1_{\mathcal{D}}(I_0)$ and its dual are understood in this sense.

We use the norms $\|\cdot\|_{\mathrm{BMO}_{\mathcal{D}},2}$ and $\|\cdot\|_{H^1_{\mathcal{D}}}$ because the constant in Theorem 2.10 is explicit in them. Moreover, $L^2(I_0) \subseteq H^1_{\mathcal{D}}(I_0)$, since by Cauchy–Schwarz on I_0, with $|I_0| = 1$, and Proposition 2.6(iii),

$$\int_{I_0} S_{\mathcal{D}} f(x)\,\mathrm{d}x \le \Big(\int_{I_0} S_{\mathcal{D}} f(x)^2\,\mathrm{d}x \Big)^{1/2} = \Big(\int_{I_0} \big| f(x) - \langle f \rangle_{I_0} \big|^2\,\mathrm{d}x \Big)^{1/2} < \infty.$$

2.5 Statement of the Duality

For $f, \varphi \in L^2(I_0)$ with $\langle \varphi \rangle_{I_0} = 0$, Proposition 2.6(iii) expands $f - \langle f \rangle_{I_0}$ in the Haar system, and orthonormality together with continuity of the inner product gives

$$\int_{I_0} f(x)\,\varphi(x)\,\mathrm{d}x = \int_{I_0} \big(f(x) - \langle f \rangle_{I_0} \big)\,\varphi(x)\,\mathrm{d}x = \sum_{I \in \mathcal{D}(I_0)} \langle f, h_I \rangle\, \langle \varphi, h_I \rangle\,, \tag{11}$$

where the first equality holds because $\int_{I_0} \varphi(x)\,\mathrm{d}x = 0$, and the series converges absolutely by Cauchy–Schwarz in ℓ^2 and Proposition 2.6(iii). This is Parseval's identity for the Haar system. Since $\mathrm{BMO}_{\mathcal{D}}(I_0)$ is defined modulo constants, we may always normalise φ so that $\langle \varphi \rangle_{I_0} = 0$. For $\varphi \in \mathrm{BMO}_{\mathcal{D}}(I_0)$ and $f \in H^1_{\mathcal{D}}(I_0)$ we write

$$\ell_\varphi(f) := \sum_{I \in \mathcal{D}(I_0)} \langle f, h_I \rangle\, \langle \varphi, h_I \rangle\,, \tag{12}$$

whenever the series converges absolutely. By (11), $\ell_\varphi(f) = \int_{I_0} f(x)\varphi(x)\,\mathrm{d}x$ for $f \in L^2(I_0)$ when $\langle \varphi \rangle_{I_0} = 0$.

Theorem 2.10 ([1, Theorem 2.1])**.** $\big(H^1_{\mathcal{D}}(I_0)\big)^* = \mathrm{BMO}_{\mathcal{D}}(I_0)$. *More precisely:*

(a) For every $\varphi \in \mathrm{BMO}_{\mathcal{D}}(I_0)$ and every $f \in H^1_{\mathcal{D}}(I_0)$,

$$\sum_{I \in \mathcal{D}(I_0)} \big|\, \langle f, h_I \rangle\, \langle \varphi, h_I \rangle\, \big| \le \sqrt{2}\, \|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2}\, \|f\|_{H^1_{\mathcal{D}}}. \tag{13}$$

Hence ℓ_φ is a bounded linear functional on $H^1_{\mathcal{D}}(I_0)$ of norm at most $\sqrt{2}\,\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2}$.

(b) Conversely, for every bounded linear functional ℓ on $H^1_{\mathcal{D}}(I_0)$ there is $\varphi \in \mathrm{BMO}_{\mathcal{D}}(I_0)$ with $\ell = \ell_\varphi$ and $\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2} \le \|\ell\|$.

Part (a) is proved by the Bellman function method in Sections 3 and 4. Part (b) is the standard direction and is sketched in Section 4.3, following [1, Lemma 2.3]. Its main step, the test on atoms, is the same as in Lecture 10 for the continuous case. The norms in [1] are twice ours by (9) and (6), so (13) is [1, Lemma 2.2], where the constant is $\sqrt{2}/4$.

Remark 2.11. Under translation and dilation of I_0, averages and $\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2}$ are invariant, while each $\langle f, h_I\rangle$ scales by $\sqrt{|I_0|}$ and $\|f\|_{H^1_{\mathcal{D}}}$ by $|I_0|$. Both sides of (13) therefore scale by $|I_0|$, so the constant $\sqrt{2}$ is independent of I_0 and the normalisation $I_0 = [0,1)$ is without loss of generality.

3 The Bellman Function

3.1 Definition and Domain

The Bellman function method proves an inequality for all f and φ at once by proving an inequality for two abstract functions on $\mathcal{D}(I_0)$. Following [1, Section 1], let $\bar{V} > 0$ and let

$$U\colon \mathcal{D}(I_0) \to [0,\infty), \qquad V\colon \mathcal{D}(I_0) \to [0,\bar{V}] \tag{14}$$

be two functions such that for every $I \in \mathcal{D}(I_0)$

$$U_{I_-} = U_{I_+} \geq U_I \qquad \text{and} \qquad V_I \geq \tfrac{1}{2}\big(V_{I_-} + V_{I_+}\big). \tag{15}$$

These are written S, M and $\bar{M}$ in [1]. By (2), the dyadic averages $I \mapsto \langle f\rangle_I$ satisfy the second condition in (15) with equality, whereas V is only required to satisfy it as an inequality. The specific U and V built from f and φ appear in Section 4.2.

Definition 3.1 (Bellman function, based on [1, Section 1])**.** Let $\Omega := [0,\infty) \times [0,\bar{V}]$. A function $\mathbb{B}\colon \Omega \to \mathbf{R}$ is called a *Bellman function* for $\bar{V}$ if it is continuous on Ω, is C^2 on $(0,\infty)\times[0,\bar{V}]$, and satisfies

$$0 \leq \mathbb{B}(x,y) \leq 2\bar{V}\sqrt{x}, \qquad -\mathbb{B}_x\mathbb{B}_y \geq \frac{\bar{V}}{2}, \qquad \mathbb{B}_{xx} \leq 0, \qquad \mathbb{B}_{yy} \geq 0, \qquad \mathbb{B}(0,y) = 0, \tag{16}$$

where the conditions involving derivatives are required for $x > 0$.

The variable x ranges over the values of U and y over the values of V, which is why the domain is the product of the two ranges in (14).

As the authors of [1] point out in their introduction, no extremal problem is posed, so $\mathbb{B}$ is not a Bellman function in the original sense of the method, whose origin [1] trace to stochastic control. Here the name instead refers to the induction on scales of Section 4.1, which [1] describe as the main feature of any Bellman function proof.

3.2 Monotonicity

Lemma 3.2 (compare [1, Section 5])**.** *Let $\mathbb{B}$ be a Bellman function for $\bar{V}$. Then for $x > 0$ and $0 \leq y \leq \bar{V}$,*

$$\mathbb{B}_x(x,y) > 0, \qquad \mathbb{B}_y(x,y) < 0.$$

Moreover, $x \mapsto \mathbb{B}_x(x,y)$ is nonincreasing and $y \mapsto \mathbb{B}_y(x,y)$ is nondecreasing.

Proof. Fix $y \in [0,\bar{V}]$. The monotonicity of $\mathbb{B}_x(\cdot,y)$ follows from $\mathbb{B}_{xx} \leq 0$, and that of $\mathbb{B}_y(x,\cdot)$ from $\mathbb{B}_{yy} \geq 0$.

Suppose $\mathbb{B}_x(x_0, y) < 0$ for some $x_0 > 0$. For $x > x_0$ the mean value theorem gives $\xi \in (x_0, x)$ with

$$\mathbb{B}(x, y) - \mathbb{B}(x_0, y) = \mathbb{B}_x(\xi, y)\,(x - x_0) \le \mathbb{B}_x(x_0, y)\,(x - x_0),$$

using $\mathbb{B}_x(\xi, y) \le \mathbb{B}_x(x_0, y)$ since $\mathbb{B}_x(\cdot, y)$ is nonincreasing and $\xi > x_0$. The right-hand side tends to $-\infty$ as $x \to \infty$ because $\mathbb{B}_x(x_0, y) < 0$, so $\mathbb{B}(x, y) < 0$ for large x, contradicting $\mathbb{B} \ge 0$. Hence $\mathbb{B}_x(x, y) \ge 0$ for all $x > 0$.

Finally, $-\mathbb{B}_x\mathbb{B}_y \ge \bar{V}/2 > 0$ shows that the product $\mathbb{B}_x\mathbb{B}_y$ is strictly negative, so neither factor vanishes and the two have opposite signs. With $\mathbb{B}_x \ge 0$ this gives $\mathbb{B}_x > 0$ and $\mathbb{B}_y < 0$. ∎

Proposition 3.3 ([1, equation (6.2)])**.** *The function*

$$\mathbb{B}(x, y) := \sqrt{x}\,\big(2\bar{V} - y\big), \qquad (x, y) \in \Omega, \tag{17}$$

is a Bellman function for $\bar{V}$.

Proof. For $x > 0$ we have

$$\mathbb{B}_x = \frac{2\bar{V} - y}{2\sqrt{x}}, \qquad \mathbb{B}_y = -\sqrt{x}, \qquad \mathbb{B}_{xx} = -\frac{2\bar{V} - y}{4x^{3/2}}, \qquad \mathbb{B}_{yy} = 0, \qquad \mathbb{B}_{xy} = -\frac{1}{2\sqrt{x}},$$

so $\mathbb{B}$ is C^2 on $(0, \infty) \times [0, \bar{V}]$ and continuous on Ω. Since $0 \le y \le \bar{V}$ we have $\bar{V} \le 2\bar{V} - y \le 2\bar{V}$ and thus

$$0 \le \bar{V}\sqrt{x} \le \mathbb{B}(x, y) \le 2\bar{V}\sqrt{x}, \qquad -\mathbb{B}_x\mathbb{B}_y = \frac{2\bar{V} - y}{2} \ge \frac{\bar{V}}{2}, \qquad \mathbb{B}_{xx} \le 0, \qquad \mathbb{B}_{yy} = 0 \ge 0,$$

and $\mathbb{B}(0, y) = 0$ directly from (17). ∎

3.3 The Key Lemma

We can now state the central inequality of the argument, which is [1, Lemma 1.1] for $d = 1$. In [1] it is indexed by interval lengths, but since $|J| = 2^{-j}$ exactly when $J \in \mathcal{D}_j(I_0)$, we index it by generations instead.

Lemma 3.4 (Key Lemma, [1, Lemma 1.1])**.** *Let U, V be as in* (14) *and* (15)*, and let $\mathbb{B}$ be a Bellman function for $\bar{V}$ in the sense of Definition 3.1. Then for every $N \ge 1$,*

$$\sum_{j=0}^{N-1} \sum_{J \in \mathcal{D}_j(I_0)} |J| \sqrt{\big(U_{J_+} - U_J\big)\Big(V_J - \tfrac{1}{2}\big(V_{J_-} + V_{J_+}\big)\Big)} \;\le\; \sqrt{2\bar{V}} \sum_{J \in \mathcal{D}_N(I_0)} |J| \sqrt{U_J}. \tag{18}$$

Both square roots are real by (15), and $U_{J_+} - U_J = U_{J_-} - U_J$. By the partition property of $\mathcal{D}_N(I_0)$ the right-hand side is an integral,

$$\sum_{J \in \mathcal{D}_N(I_0)} |J| \sqrt{U_J} = \int_{I_0} \sum_{J \in \mathcal{D}_N(I_0)} \sqrt{U_J}\, \chi_J(x)\, \mathrm{d}x. \tag{19}$$

Remark 3.5 (Why this $\mathbb{B}$)**.** The function (17) is chosen to make the constant in (18) as small as possible, as follows [1, Section 6]. Replace the first two conditions in (16) by $0 \le \mathbb{B} \le a_1\sqrt{x}$ and $-\mathbb{B}_x\mathbb{B}_y \ge a_2$ with constants $a_1, a_2 > 0$. The proof in Section 4.1 still works, and the constant $\sqrt{2\bar{V}}$ in (18) becomes a multiple of $a_1/\sqrt{a_2}$. So we want a_1 small and a_2 large. The simplest candidates are $\mathbb{B}(x,y) = \sqrt{x}\,(A-y)$ with $A > \bar{V}$ since the factor $\sqrt{x}$ matches the size bound, and $A-y$ is the simplest decreasing function of y with $\mathbb{B}_{yy} = 0$. For these, $a_1 = A$ and $-\mathbb{B}_x\mathbb{B}_y = \frac{1}{2}(A-y) \ge \frac{1}{2}(A-\bar{V}) = a_2$, so

$$\frac{a_1}{\sqrt{a_2}} = \frac{\sqrt{2}\,A}{\sqrt{A-\bar{V}}},$$

which is smallest at $A = 2\bar{V}$. This gives (17). The constant is unlikely to be sharp [1, Remark 6.1], since the two conditions are attained at different values of y (the size bound at $y = 0$, the product condition at $y = \bar{V}$) while the proof uses both at once.

4 Proof of the Duality

This section proves Theorem 2.10(a), following [1, Sections 2 and 5].

4.1 Bellman Induction on the Dyadic Tree

The proof has two parts: an inequality comparing $\mathbb{B}$ on an interval with $\mathbb{B}$ on its two children, and a telescoping sum of that inequality over the generations.

Lemma 4.1. *Let $U, V, \mathbb{B}$ be as in Lemma 3.4. Then for every $J \in \mathcal{D}(I_0)$,*

$$\tfrac{1}{2}\mathbb{B}\big(U_{J_-}, V_{J_-}\big) + \tfrac{1}{2}\mathbb{B}\big(U_{J_+}, V_{J_+}\big) \;\ge\; \mathbb{B}\big(U_J, V_J\big) + \sqrt{2\bar{V}}\,\sqrt{\big(U_{J_+} - U_J\big)\Big(V_J - \tfrac{1}{2}\big(V_{J_-} + V_{J_+}\big)\Big)}. \tag{20}$$

Proof. Fix J and abbreviate

$$u := U_J, \qquad u' := U_{J_-} = U_{J_+}, \qquad v := V_J, \qquad v_\pm := V_{J_\pm}, \qquad v_0 := \tfrac{1}{2}(v_- + v_+),$$

so that by (15) $0 \le u \le u'$ and $0 \le v_0 \le v \le \bar{V}$, and (20) reads

$$\tfrac{1}{2}\mathbb{B}(u', v_-) + \tfrac{1}{2}\mathbb{B}(u', v_+) \;\ge\; \mathbb{B}(u, v) + \sqrt{2\bar{V}}\,\sqrt{(u'-u)(v-v_0)}. \tag{21}$$

If $u' = 0$ then $u = 0$, both sides of (21) vanish by $\mathbb{B}(0,\cdot) = 0$, and there is nothing to prove. Assume now that $u' > 0$, so that Lemma 3.2 applies at $x = u'$.

Step 1 (convexity in y). Since $\mathbb{B}_{yy} \ge 0$, the function $y \mapsto \mathbb{B}(u', y)$ is convex on $[0, \bar{V}]$, and Jensen's inequality with two points gives

$$\tfrac{1}{2}\mathbb{B}(u', v_-) + \tfrac{1}{2}\mathbb{B}(u', v_+) \;\ge\; \mathbb{B}(u', v_0).$$

Step 2 (moving y). By the mean value theorem in y there is $\eta \in [v_0, v]$ with

$$\mathbb{B}(u', v_0) \;=\; \mathbb{B}(u', v) - \mathbb{B}_y(u', \eta)\,(v - v_0) \;\ge\; \mathbb{B}(u', v) - \mathbb{B}_y(u', v)\,(v - v_0),$$

where the inequality uses $v - v_0 \ge 0$ and $-\mathbb{B}_y(u', \eta) \ge -\mathbb{B}_y(u', v)$, since $\mathbb{B}_y(u', \cdot)$ is nondecreasing (Lemma 3.2) and $\eta \le v$.

Step 3 (moving x). We claim

$$\mathbb{B}(u', v) \;\ge\; \mathbb{B}(u, v) + \mathbb{B}_x(u', v)\,(u' - u). \tag{22}$$

If $u > 0$, the mean value theorem in x gives $\xi \in [u, u']$ with $\mathbb{B}(u', v) - \mathbb{B}(u, v) = \mathbb{B}_x(\xi, v)(u' - u)$, and $\mathbb{B}_x(\xi, v) \geq \mathbb{B}_x(u', v)$ since $\mathbb{B}_x(\cdot, v)$ is nonincreasing and $\xi \leq u'$. If $u = 0$, apply this with $s \in (0, u')$ in place of u to obtain $\mathbb{B}(u', v) - \mathbb{B}(s, v) \geq \mathbb{B}_x(u', v)(u' - s)$, and let $s \to 0$ using the continuity of $\mathbb{B}$ and $\mathbb{B}(0, v) = 0$. This limiting argument is the reason only continuity of $\mathbb{B}$ is required at $x = 0$ in Definition 3.1.

Step 4 (combination). Steps 1, 2 and 3 give, in that order,

$$\begin{aligned} \tfrac{1}{2}\mathbb{B}(u', v_-) + \tfrac{1}{2}\mathbb{B}(u', v_+) \;&\geq\; \mathbb{B}(u', v_0) \;\geq\; \mathbb{B}(u', v) - \mathbb{B}_y(u', v)\,(v - v_0) \\ &\geq\; \mathbb{B}(u, v) + \underbrace{\mathbb{B}_x(u', v)\,(u' - u)}_{=:\,a} + \underbrace{\big(-\mathbb{B}_y(u', v)\big)(v - v_0)}_{=:\,b}, \end{aligned}$$

where $a, b \geq 0$ by Lemma 3.2 and (15). By the arithmetic–geometric mean inequality and the mixed condition in (16) at the point (u', v),

$$\begin{aligned} a + b \;\geq\; 2\sqrt{ab} \;&=\; 2\sqrt{\big(-\mathbb{B}_x(u', v)\mathbb{B}_y(u', v)\big)(u' - u)(v - v_0)} \\ &\geq\; 2\sqrt{\tfrac{\bar{V}}{2}}\,\sqrt{(u' - u)(v - v_0)} \;=\; \sqrt{2\bar{V}}\,\sqrt{(u' - u)(v - v_0)}. \end{aligned}$$

Hence $\frac{1}{2}\mathbb{B}(u', v_-) + \frac{1}{2}\mathbb{B}(u', v_+) \geq \mathbb{B}(u, v) + \sqrt{2\bar{V}}\,\sqrt{(u' - u)(v - v_0)}$, which is (21). ∎

Now we move to the proof of the Key Lemma.

Proof of Lemma 3.4. For $0 \leq j \leq N$ set

$$G_j := \sum_{J \in \mathcal{D}_j(I_0)} |J|\,\mathbb{B}\big(U_J, V_J\big), \qquad F_j := \sum_{J \in \mathcal{D}_j(I_0)} |J| \sqrt{\big(U_{J_+} - U_J\big)\Big(V_J - \tfrac{1}{2}\big(V_{J_-} + V_{J_+}\big)\Big)},$$

so that the left-hand side of (18) is $\sum_{j=0}^{N-1} F_j$. The intervals of generation $j+1$ are exactly the children of the intervals of generation j, each child having length $\frac{1}{2}|J|$, so

$$\begin{aligned} G_{j+1} &= \sum_{J \in \mathcal{D}_j(I_0)} \Big(|J_-|\,\mathbb{B}\big(U_{J_-}, V_{J_-}\big) + |J_+|\,\mathbb{B}\big(U_{J_+}, V_{J_+}\big)\Big) \\ &= \sum_{J \in \mathcal{D}_j(I_0)} |J| \Big(\tfrac{1}{2}\mathbb{B}\big(U_{J_-}, V_{J_-}\big) + \tfrac{1}{2}\mathbb{B}\big(U_{J_+}, V_{J_+}\big)\Big) \\ &\overset{(20)}{\geq} \sum_{J \in \mathcal{D}_j(I_0)} |J| \Big(\mathbb{B}\big(U_J, V_J\big) + \sqrt{2\bar{V}}\,\sqrt{\big(U_{J_+} - U_J\big)\Big(V_J - \tfrac{1}{2}\big(V_{J_-} + V_{J_+}\big)\Big)}\Big) \\ &= G_j + \sqrt{2\bar{V}}\,F_j. \end{aligned}$$

Iterating from $j = 0$ to $j = N - 1$,

$$G_N \;\geq\; G_0 + \sqrt{2\bar{V}} \sum_{j=0}^{N-1} F_j \;\geq\; \sqrt{2\bar{V}} \sum_{j=0}^{N-1} F_j,$$

since $G_0 = |I_0|\,\mathbb{B}(U_{I_0}, V_{I_0}) \geq 0$ by the size bound in (16). The same size bound gives $\mathbb{B}(U_J, V_J) \leq 2\bar{V}\sqrt{U_J}$, hence

$$\sqrt{2\bar{V}} \sum_{j=0}^{N-1} F_j \;\leq\; G_N \;\leq\; 2\bar{V} \sum_{J \in \mathcal{D}_N(I_0)} |J|\sqrt{U_J},$$

and dividing by $\sqrt{2\bar{V}}$ gives (18). ∎

4.2 The Key Estimate

Let $f \in L^1(I_0)$ and $\varphi \in \mathrm{BMO}_{\mathcal{D}}(I_0)$ with $\langle \varphi \rangle_{I_0} = 0$. We may assume $\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2} > 0$, since otherwise $\varphi = 0$ and (13) is trivial. This is needed because Definition 3.1 requires $\bar{V} > 0$. Following [1, proof of Lemma 2.2], define for $J \in \mathcal{D}(I_0)$

$$U_J := \sum_{I \supsetneq J} \big(\langle f \rangle_{I_-} - \langle f \rangle_{I_+}\big)^2, \qquad V_J := \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} |I| \big(\langle \varphi \rangle_{I_-} - \langle \varphi \rangle_{I_+}\big)^2, \qquad \bar{V} := 4\,\|\varphi\|^2_{\mathrm{BMO}_{\mathcal{D}},2}. \tag{23}$$

Since Proposition 2.6(i) holds for $f \in L^1(I_0)$, so does (5), and for $J \in \mathcal{D}_j(I_0)$ and $x \in J$,

$$U_J = 4\, S_{\mathcal{D},j} f(x)^2, \qquad \text{in particular } U_{I_0} = 4\, S_{\mathcal{D},0} f(x)^2 = 0. \tag{24}$$

By Proposition 2.6(iii) on J and (9),

$$V_J = \frac{4}{|J|} \int_J \big|\varphi(x) - \langle \varphi \rangle_J\big|^2 \,\mathrm{d}x, \qquad 0 \le V_J \le \bar{V}. \tag{25}$$

So U and V take values in $[0, \infty)$ and $[0, \bar{V}]$ as required in (14).

Lemma 4.2. *U and V satisfy* (15)*, with*

$$\begin{aligned} U_{J_\pm} - U_J &= \big(\langle f \rangle_{J_-} - \langle f \rangle_{J_+}\big)^2 = \frac{4\,\langle f, h_J \rangle^2}{|J|}, \\ V_J - \tfrac{1}{2}\big(V_{J_-} + V_{J_+}\big) &= \big(\langle \varphi \rangle_{J_-} - \langle \varphi \rangle_{J_+}\big)^2 = \frac{4\,\langle \varphi, h_J \rangle^2}{|J|}. \end{aligned} \tag{26}$$

Proof. By (1) the strict ancestors of J_- are those of J together with J itself, and likewise for J_+. Hence

$$U_{J_-} = U_{J_+} = U_J + \big(\langle f \rangle_{J_-} - \langle f \rangle_{J_+}\big)^2,$$

which gives $U_{J_-} = U_{J_+} \ge U_J$, the first condition in (15), and the first identity in (26). For V, every $I \in \mathcal{D}(J)$ is either J itself or, having $|I| \le \frac{1}{2}|J|$, contained in exactly one of J_-, J_+ (Lecture 4, Exercise). Hence every $I \in \mathcal{D}(J)$ belongs to exactly one of $\{J\}$, $\mathcal{D}(J_-)$, $\mathcal{D}(J_+)$, and splitting the sum in (23) accordingly,

$$|J|\, V_J = |J| \big(\langle \varphi \rangle_{J_-} - \langle \varphi \rangle_{J_+}\big)^2 + |J_-|\, V_{J_-} + |J_+|\, V_{J_+}.$$

Dividing by $|J|$ and using $|J_\pm| = \frac{1}{2}|J|$,

$$V_J = \big(\langle \varphi \rangle_{J_-} - \langle \varphi \rangle_{J_+}\big)^2 + \tfrac{1}{2}\big(V_{J_-} + V_{J_+}\big),$$

which gives $V_J \ge \frac{1}{2}(V_{J_-} + V_{J_+})$, the second condition in (15), and the second identity in (26). The Haar forms in (26) follow from Definition 2.5. ∎

Proposition 4.3 (Key Estimate)**.** *For every $N \ge 1$,*

$$\sum_{j=0}^{N-1} \sum_{J \in \mathcal{D}_j(I_0)} \big|\, \langle f, h_J \rangle\, \langle \varphi, h_J \rangle \,\big| \;\le\; \sqrt{2}\, \|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2} \int_{I_0} S_{\mathcal{D},N} f(x)\,\mathrm{d}x. \tag{27}$$

Proof. By Proposition 3.3 there is a Bellman function for $\bar{V}$, and by Lemma 4.2 the pair U, V satisfies the hypotheses of Lemma 3.4. We compute both sides of (18). On the left, by (26),

$$|J|\sqrt{\big(U_{J_+} - U_J\big)\Big(V_J - \tfrac{1}{2}\big(V_{J_-} + V_{J_+}\big)\Big)} = |J|\sqrt{\frac{4\,\langle f, h_J\rangle^2}{|J|}\cdot\frac{4\,\langle \varphi, h_J\rangle^2}{|J|}} = 4\,\big|\,\langle f, h_J\rangle\,\langle \varphi, h_J\rangle\,\big|.$$

On the right, by (24) $\sqrt{U_J} = 2\,S_{\mathcal{D},N} f(x)$ for $x \in J \in \mathcal{D}_N(I_0)$, so by (19)

$$\begin{aligned}\sqrt{2\bar{V}} \sum_{J\in\mathcal{D}_N(I_0)} |J|\sqrt{U_J} &= \sqrt{8}\,\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2}\cdot 2\int_{I_0} S_{\mathcal{D},N} f(x)\,\mathrm{d}x \\ &= 4\sqrt{2}\,\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2}\int_{I_0} S_{\mathcal{D},N} f(x)\,\mathrm{d}x.\end{aligned}$$

Inserting both into (18),

$$4\sum_{j=0}^{N-1}\sum_{J\in\mathcal{D}_j(I_0)} \big|\,\langle f, h_J\rangle\,\langle \varphi, h_J\rangle\,\big| \;\le\; 4\sqrt{2}\,\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2}\int_{I_0} S_{\mathcal{D},N} f(x)\,\mathrm{d}x,$$

and dividing by 4 gives (27). ∎

4.3 Conclusion of the Proof

Proof of Theorem 2.10(a). Let $f \in H^1_{\mathcal{D}}(I_0)$ and $\varphi \in \mathrm{BMO}_{\mathcal{D}}(I_0)$, normalised by $\langle\varphi\rangle_{I_0} = 0$. Let $N \to \infty$ in (27). On the left, the partial sums over generations $0, \dots, N-1$ have nonnegative terms and increase to the sum over all of $\mathcal{D}(I_0)$. On the right, $S_{\mathcal{D},N} f$ increases pointwise to $S_{\mathcal{D}} f$ by (4), so as $N \to \infty$ by the monotone convergence theorem

$$\int_{I_0} S_{\mathcal{D},N} f(x)\,\mathrm{d}x \;\to\; \int_{I_0} S_{\mathcal{D}} f(x)\,\mathrm{d}x = \|f\|_{H^1_{\mathcal{D}}}.$$

This is (13). In particular the series (12) converges absolutely, ℓ_φ is linear, and $|\ell_\varphi(f)| \le \sqrt{2}\,\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2}\|f\|_{H^1_{\mathcal{D}}}$. ∎

Sketch of the proof of Theorem 2.10(b). We follow the structure of [1, Lemma 2.3], with the L^2 atoms of Lecture 10 in place of the L^∞ atoms used there. Let ℓ be a bounded linear functional on $H^1_{\mathcal{D}}(I_0)$ and write $L^2_0(I_0) := \{g \in L^2(I_0) : \langle g\rangle_{I_0} = 0\}$. By the inclusion at the end of Section 2.4, $\|g\|_{H^1_{\mathcal{D}}} \le \|g\|_{L^2}$ for $g \in L^2_0(I_0)$, so ℓ restricted to $L^2_0(I_0)$ is bounded in the L^2 norm, and the Riesz representation theorem gives $\varphi \in L^2_0(I_0)$ with

$$\ell(g) = \int_{I_0} g(x)\,\varphi(x)\,\mathrm{d}x \qquad \text{for all } g \in L^2_0(I_0). \tag{28}$$

To see that $\varphi \in \mathrm{BMO}_{\mathcal{D}}(I_0)$ we test (28) on dyadic atoms, as in Lecture 10. Fix $J \in \mathcal{D}(I_0)$ and let $a \in L^2(I_0)$ be supported in J with $\int_J a(x)\,\mathrm{d}x = 0$ and $\|a\|_{L^2} \le |J|^{-1/2}$. Since a has mean zero on J, $\mathbb{D}_k a$ vanishes for k up to the generation of J and is supported in J for all larger k (Lecture 15, Steps 1 and 2 in the proof of Theorem 3), so $S_{\mathcal{D}} a$ is supported in J, and by Cauchy–Schwarz on J and Proposition 2.6(iii) on I_0 with $\langle a\rangle_{I_0} = 0$,

$$\|a\|_{H^1_{\mathcal{D}}} = \int_J S_{\mathcal{D}} a(x)\,\mathrm{d}x \le |J|^{1/2}\Big(\int_J S_{\mathcal{D}} a(x)^2\,\mathrm{d}x\Big)^{1/2} = |J|^{1/2}\,\|a\|_{L^2} \le 1.$$

Hence

$$\Big| \int_J \big(\varphi(x) - \langle\varphi\rangle_J\big)\, a(x)\, \mathrm{d}x \Big| = \Big| \int_{I_0} \varphi(x)\, a(x)\, \mathrm{d}x \Big| = |\ell(a)| \le \|\ell\|.$$

As in Lecture 10, any $a \in L^2(J)$ with $\|a\|_{L^2} \le |J|^{-1/2}$ can be replaced by $a - \langle a\rangle_J \chi_J$ without changing the left-hand side or increasing the L^2 norm, so the bound holds for all such a, and by duality in $L^2(J)$

$$\Big(\frac{1}{|J|}\int_J \big|\varphi(x) - \langle\varphi\rangle_J\big|^2\, \mathrm{d}x\Big)^{1/2} = \sup_{\|a\|_{L^2(J)} \le |J|^{-1/2}} \Big| \int_J \big(\varphi(x) - \langle\varphi\rangle_J\big)\, a(x)\, \mathrm{d}x \Big| \le \|\ell\|.$$

Taking the supremum over J gives $\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2} \le \|\ell\|$. Finally, ℓ and ℓ_φ are both bounded on $H^1_{\mathcal{D}}(I_0)$, both vanish on constants by Remark 2.9, and agree on $L^2_0(I_0)$ by (28) and (11), so they agree on the closure of $L^2_0(I_0)$ modulo constants. That this closure is all of $H^1_{\mathcal{D}}(I_0)$ is shown in [1, Section 2] by truncating the Haar expansion of $f - \langle f\rangle_{I_0}$ at generation N and letting $N \to \infty$. We omit the details.

Remark 4.4. Sections 3 and 4 use only that $\mathcal{D}(I_0)$ is a binary tree whose children have half the length of the parent, together with the inequality (20). The same Bellman function (17) handles the dyadic lattice rooted in a cube in $\mathbf{R}^d$, where each cube has 2^d children and the two-point Jensen step becomes a 2^d-point one. It also handles the continuous case on the circle $\mathbf{T}$ [1, Section 3], there U_J and V_J are built from area integrals of the harmonic extensions of f and φ over dyadic Carleson boxes, the same Key Lemma is applied, and Green's formula bounds the limit of the left-hand side from below by a constant multiple of $\int_{\mathbf{T}} \varphi \bar{f}$. The multi-dimensional continuous case is [1, Section 4].

Bibliography

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