Duality between Dyadic H^1 and Dyadic BMO via Bellman Function Techniques

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MATP32 Harmonic Analysis, November 2026

What Is a Dual Space?

Definition

The *dual space* X^* of a normed space X consists of all *bounded linear functionals*

$$\ell \colon X \to \mathbf{R} \text{ linear}, \qquad |\ell(f)| \leq C \, \|f\|_X \text{ for all } f \in X.$$

Model case: $\left(L^p\right)^* = L^q$ for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Two things:

1. *An inequality* (Hölder): $\left| \int fg \, \mathrm{d}x \right| \leq \|g\|_{L^q} \, \|f\|_{L^p}$, so every $g \in L^q$ is a bounded functional on L^p.
2. *A representation:* every bounded functional has this form.

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At $p = 1$: $\left(L^1\right)^* = L^\infty$, and a smaller space has *more* functionals.

A Theorem We Already Know

Shrink L^1 to H^1, and the dual grows from L^∞ to BMO:

$$(H^1(\mathbf{R}^d))^* = \mathrm{BMO}(\mathbf{R}^d).$$

Our proof in Lecture 10: atoms, the John–Nirenberg inequality, the Calderón–Zygmund decomposition.

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Question: how can a function of two numbers know about every f and every φ?

The Dyadic Tree

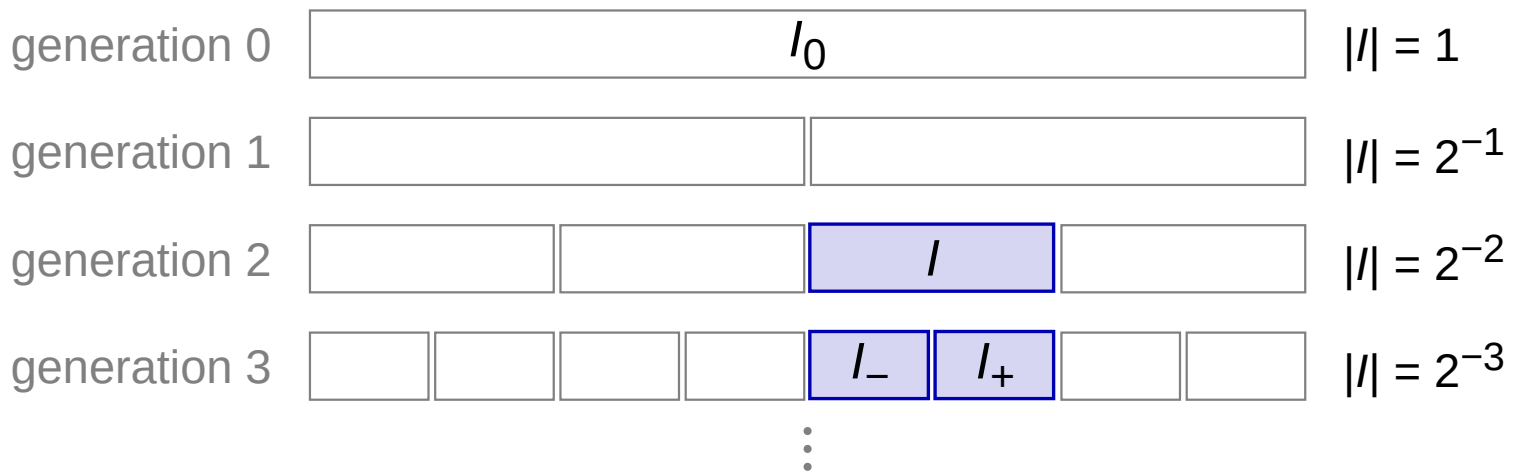


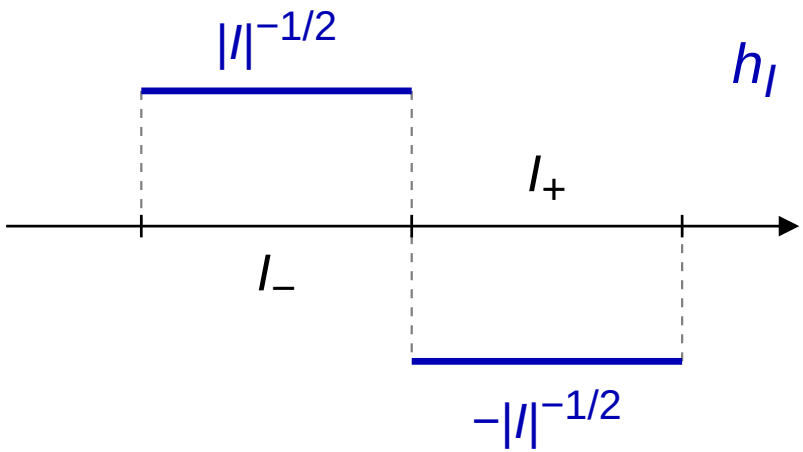
Figure 1: The dyadic intervals $\mathcal{D}(I_0)$ of $I_0 = [0, 1)$.

Every I has two *children* I_- and I_+, and the averages satisfy

$$\langle f \rangle_I = \tfrac{1}{2}\big(\langle f \rangle_{I_-} + \langle f \rangle_{I_+}\big), \quad \text{where} \quad \langle f \rangle_I := \frac{1}{|I|} \int_I f(x)\,\mathrm{d}x.$$

This is the *martingale property* of Lecture 13.

Haar Functions



For $I \in \mathcal{D}(I_0)$, as in Lecture 16,

$$h_I := |I|^{-1/2}(\chi_{I_-} - \chi_{I_+}).$$

The h_I are orthonormal in $L^2(I_0)$.

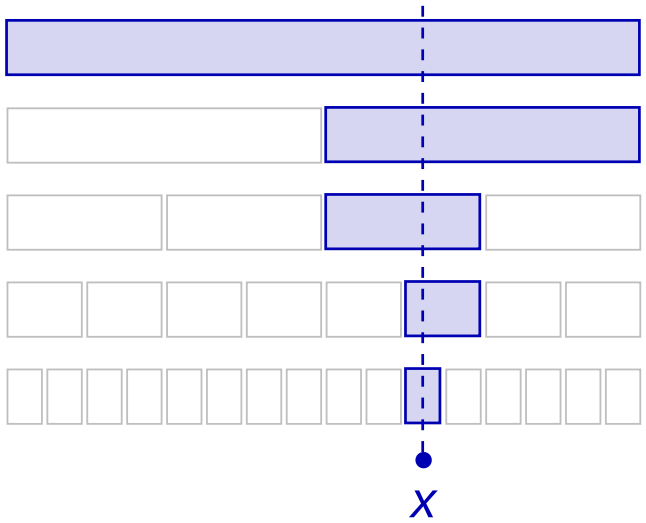
The *Haar coefficient* of f at I is the jump of the average between the children:

$$\langle f, h_I \rangle = \frac{\sqrt{|I|}}{2}(\langle f \rangle_{I_-} - \langle f \rangle_{I_+}).$$

One coefficient for every interval of the tree.

Two Ways to Measure Haar Coefficients

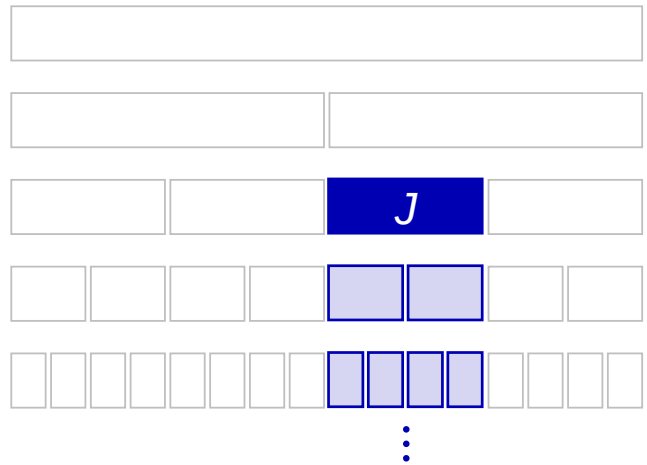
Dyadic H^1: along a branch



$$S_{\mathcal{D}} f(x)^2 = \sum_{I \ni x} \frac{\langle f, h_I \rangle^2}{|I|}$$

$$\|f\|_{H^1_{\mathcal{D}}} := \int_{I_0} S_{\mathcal{D}} f(x)\, \mathrm{d}x$$

Dyadic BMO: over a subtree



$$\|\varphi\|^2_{\mathrm{BMO}_{\mathcal{D}},2} := \sup_J \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \langle \varphi, h_I \rangle^2$$

This is $\sup_J \frac{1}{|J|} \int_J |\varphi - \langle \varphi \rangle_J|^2 \, \mathrm{d}x$.

The Duality Theorem

Theorem ([1, Theorem 2.1])

$\left(H^1_{\mathcal{D}}(I_0)\right)^* = \mathrm{BMO}_{\mathcal{D}}(I_0)$. *The hard direction is the estimate*

$$\sum_{I \in \mathcal{D}(I_0)} \left|\langle f, h_I\rangle \langle \varphi, h_I\rangle\right| \leq \sqrt{2}\, \|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2}\, \|f\|_{H^1_{\mathcal{D}}}$$

for all $f \in H^1_{\mathcal{D}}(I_0)$ *and* $\varphi \in \mathrm{BMO}_{\mathcal{D}}(I_0)$.

1. For $f \in L^2(I_0)$ and $\langle \varphi \rangle_{I_0} = 0$, the sum without absolute values is $\int_{I_0} f\varphi \, \mathrm{d}x$.
2. The pairing is written *interval by interval*. Each term lives at one node of the tree.

Why Is It Hard?

First attempt: Cauchy–Schwarz and Parseval give

$$\sum_I \left|\langle f, h_I\rangle \langle \varphi, h_I\rangle\right| \le \left\|f - \langle f\rangle_{I_0}\right\|_{L^2} \left\|\varphi - \langle \varphi\rangle_{I_0}\right\|_{L^2}.$$

True, but it needs $f \in L^2(I_0)$. The theorem makes a *trade*:

1. *less of* f: only $S_{\mathcal{D}} f \in L^1$, a condition on branches,
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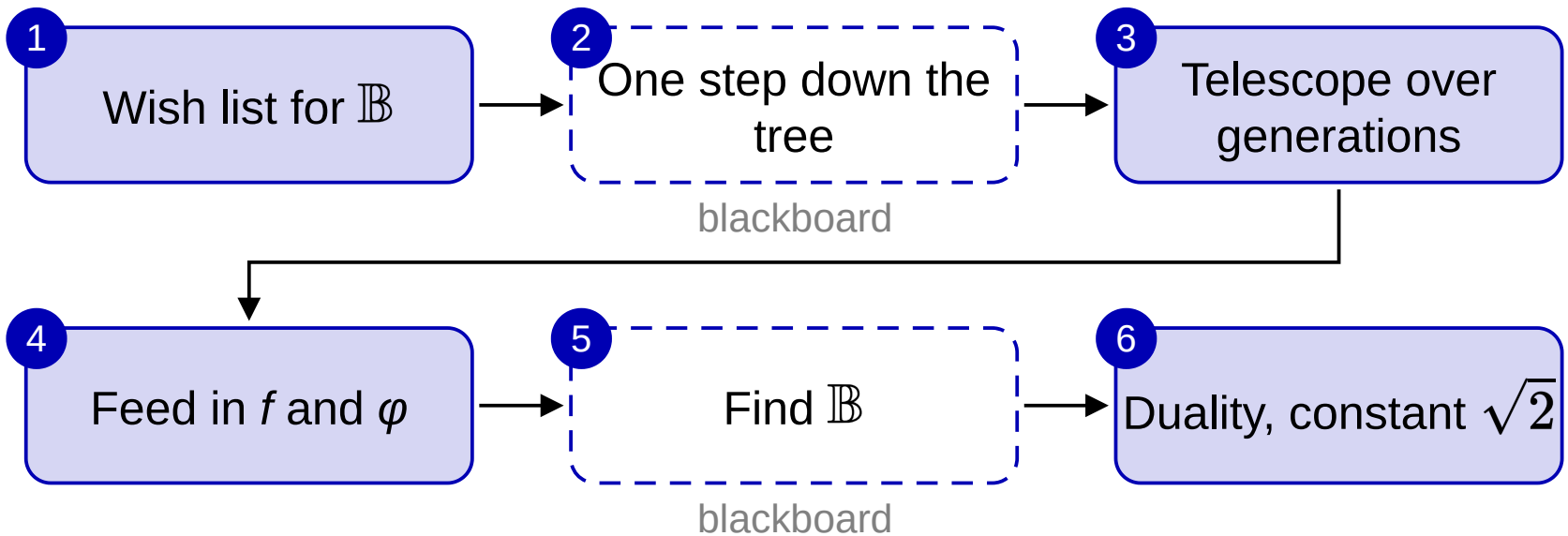
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We must pair a branch norm with a subtree norm, one scale at a time.

The Plan

Idea of a Bellman proof: never look at the whole tree. Control *one step*, from a parent to its two children, and let one function carry everything else.



What Is a Bellman Function?

1. *Origin:* R. Bellman's dynamic programming (1950s). To control a process over many steps, control *one step* and let a *value function* remember the rest.
2. *In analysis:* Nazarov, Treil and Volberg (1990s). Burkholder's sharp martingale inequalities (1984) fit the same framework.
3. *For us:* the process walks down the dyadic tree, from J to J_- or J_+. At each step the function must *gain* at least the term we want to sum.

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Remark: we pose no extremal problem, so we do not find *the* Bellman function. What survives of the method is the *induction on scales* [1].

The Bank Account

Generation j charges a *toll* $T_j \geq 0$. We look for a *balance* $G_j \geq 0$ with

$$G_{j+1} \geq G_j + T_j.$$

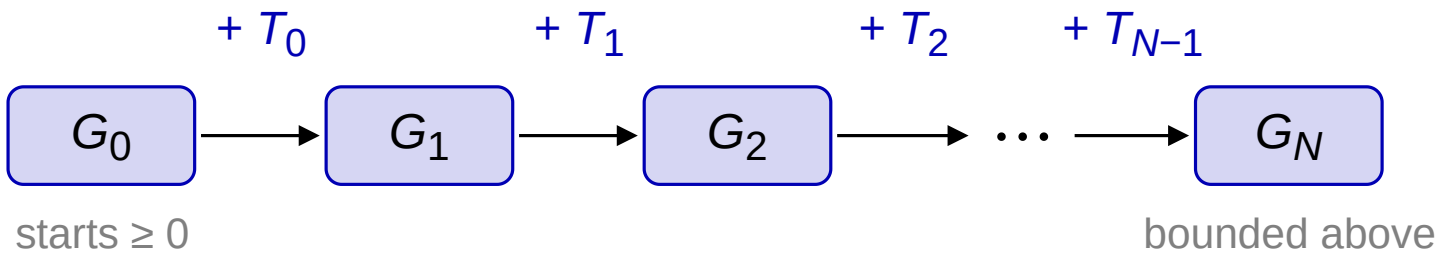


Figure 2: The balance grows by at least the toll at every step.

Then all the tolls together are paid by the final balance:

$$\sum_{j=0}^{N-1} T_j \leq G_N - G_0 \leq G_N.$$

We never estimate the sum, only one step.

Two Numbers per Interval

Attach to every $J \in \mathcal{D}(I_0)$ numbers $U_J \in [0, \infty)$ and $V_J \in [0, \bar{V}]$ with

$$U_{J_-} = U_{J_+} \geq U_J \qquad \text{and} \qquad V_J \geq \tfrac{1}{2}(V_{J_-} + V_{J_+}).$$

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Toll at J: the geometric mean of the two changes,

$$t_J := \sqrt{(U_{J_+} - U_J)\Big(V_J - \tfrac{1}{2}(V_{J_-} + V_{J_+})\Big)}\,.$$

Balance and toll of generation j:

$$G_j := \sum_{J \in \mathcal{D}_j(I_0)} |J|\, \mathbb{B}(U_J, V_J), \qquad T_j := \sqrt{2\bar{V}} \sum_{J \in \mathcal{D}_j(I_0)} |J|\, t_J.$$

The Wish List

Definition

A function $\mathbb{B}\colon [0,\infty) \times [0,\bar{V}] \to \mathbf{R}$, continuous and C^2 for $x > 0$, is a *Bellman function* for $\bar{V}$ if

size	$0 \le \mathbb{B}(x,y) \le 2\bar{V}\sqrt{x}$	the balance is bounded
shape	$\mathbb{B}_{xx} \le 0$ and $\mathbb{B}_{yy} \ge 0$	compare parent and children
gain	$-\mathbb{B}_x\,\mathbb{B}_y \ge \dfrac{\bar{V}}{2}$	every step pays the toll

Consequences: $\mathbb{B}(0,y) = 0$, $\mathbb{B}_x > 0$ and $\mathbb{B}_y < 0$.

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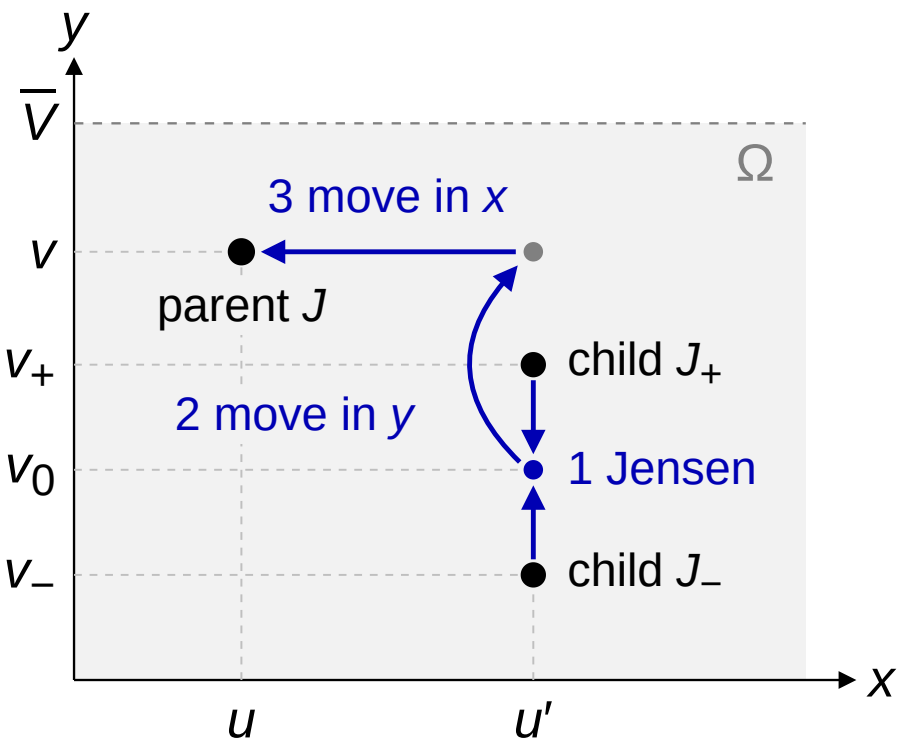
Does such a function exist? Hold that thought.

One Step Down the Tree

Lemma

Let $\mathbb{B}$ be a Bellman function for $\bar{V}$. Then for every $J \in \mathcal{D}(I_0)$

$$\tfrac{1}{2}\mathbb{B}(u', v_-) + \tfrac{1}{2}\mathbb{B}(u', v_+) \geq \mathbb{B}(u, v) + \sqrt{2\bar{V}}\sqrt{(u' - u)(v - v_0)}.$$



$u := U_J, \quad u' := U_{J_-} = U_{J_+},$
$v := V_J, \quad v_\pm := V_{J_\pm},$
$v_0 := \frac{1}{2}(v_- + v_+).$

1. $\mathbb{B}_{yy} \geq 0$
2. $\mathbb{B}_y$ nondecreasing in y
3. $\mathbb{B}_x$ nonincreasing in x
4. $a + b \geq 2\sqrt{ab}$ and the gain

→ *blackboard: proof*

Telescoping: the Key Lemma

Multiply the one-step inequality by $|J|$ and sum over $J \in \mathcal{D}_j(I_0)$. Since $|J_\pm| = \frac{1}{2}|J|$, this is exactly

$$G_{j+1} \geq G_j + T_j.$$

The bank account gives $\sum_{j<N} T_j \leq G_N$, and the *size* condition bounds G_N.

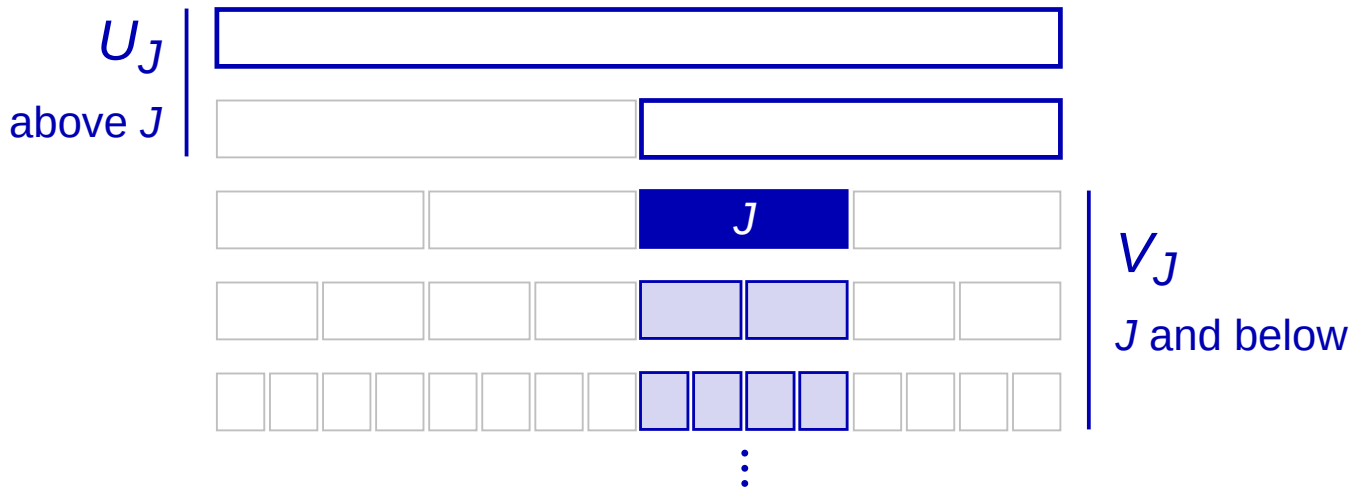
Lemma (Key Lemma, [1, Lemma 1.1])

For every $N \geq 1$,

$$\sum_{j=0}^{N-1} \sum_{J \in \mathcal{D}_j(I_0)} |J|\, t_J \;\leq\; \sqrt{2\bar{V}} \sum_{J \in \mathcal{D}_N(I_0)} |J| \sqrt{U_J}.$$

Still abstract: no f and no φ yet.

Feeding in f and φ



$$U_J := \sum_{I \supsetneq J} \left(\langle f \rangle_{I_-} - \langle f \rangle_{I_+}\right)^2 = 4\, S_{\mathcal{D},j} f(x)^2, \qquad x \in J \in \mathcal{D}_j(I_0),$$

$$V_J := \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} |I| \left(\langle \varphi \rangle_{I_-} - \langle \varphi \rangle_{I_+}\right)^2 \leq 4 \, \|\varphi\|^2_{\mathrm{BMO}_{\mathcal{D}},2} =: \bar{V}.$$

U collects f on the branch above J, and V collects φ on the subtree below.

The Two Changes Meet at J

Going from J to its children,

$$U_{J_\pm} - U_J = \frac{4\langle f, h_J\rangle^2}{|J|},$$
$$V_J - \tfrac{1}{2}(V_{J_-} + V_{J_+}) = \frac{4\langle \varphi, h_J\rangle^2}{|J|}.$$

So U *gains* the Haar coefficient of f at J, and V *loses* that of φ.

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The toll at J is their product:

$$|J|\; t_J = 4\,\big|\langle f, h_J\rangle\,\langle \varphi, h_J\rangle\big|.$$

The Key Estimate

Insert U and V into the Key Lemma.

Proposition

For every $N \geq 1$,

$$\sum_{j=0}^{N-1} \sum_{J \in \mathcal{D}_j(I_0)} \left| \langle f, h_J \rangle \langle \varphi, h_J \rangle \right| \leq \sqrt{2} \, \|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2} \int_{I_0} S_{\mathcal{D},N} f(x) \, \mathrm{d}x.$$

Let $N \to \infty$. By the monotone convergence theorem the right-hand side tends to $\sqrt{2} \, \|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2} \, \|f\|_{H^1_{\mathcal{D}}}$.

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This is the theorem . . . *provided a Bellman function exists.*

The Function

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Reading it with our U and V:

$$\mathbb{B}(U_J, V_J) = 2\, S_{\mathcal{D},j} f(x) \cdot (2\bar{V} - V_J).$$

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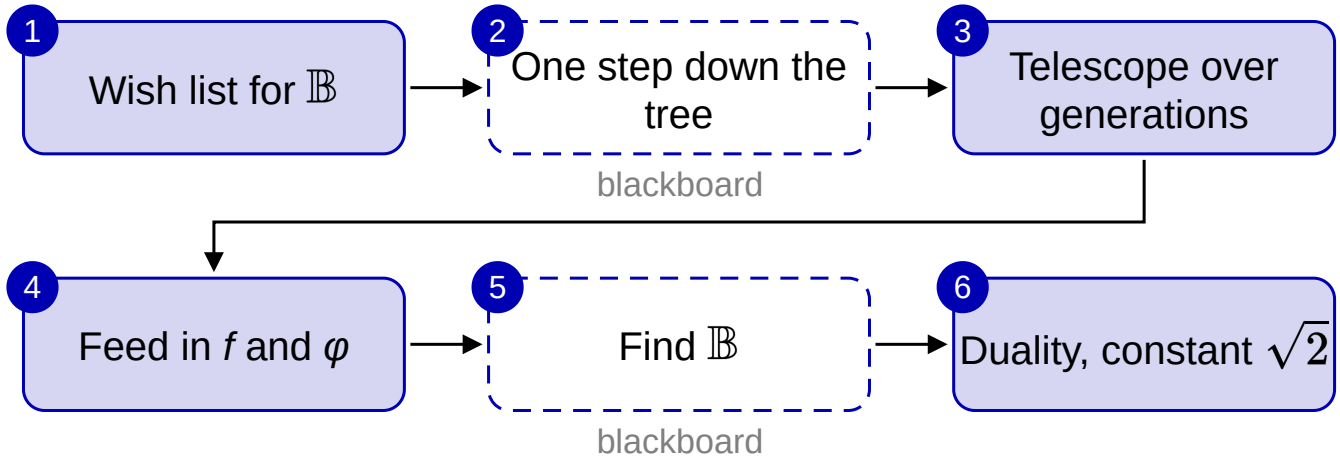
Why $2\bar{V}$? Among all $\sqrt{x}\,(A - y)$ with $A > \bar{V}$ it gives the smallest constant.

$\rightarrow$ *blackboard: check the wish list, conclude the proof*

What Did We Gain?

1. *An explicit constant* $\sqrt{2}$, which is unlikely to be sharp [1, Remark 6.1].
2. *An elementary proof.* No atoms, no Calderón–Zygmund decomposition, no John–Nirenberg inequality. Only Jensen, the mean value theorem and $a + b \geq 2\sqrt{ab}$.
3. *Only the tree matters.* The same $\mathbb{B}$ handles dyadic cubes in $\mathbf{R}^d$ and the continuous case on the circle [1].
4. *The converse* is standard, as in Lecture 10: the Riesz representation theorem in L^2 and a test on atoms give $\|\varphi\|_{\mathrm{BMO}_{\mathcal{D}},2} \leq \|\ell\|$.

Summary



1. f lives on *branches*, φ lives on *subtrees*, and the Haar coefficient at J is where they meet.
2. One step is Jensen, the mean value theorem and $a + b \geq 2\sqrt{ab}$. All steps are a telescoping sum.

A function of two numbers knows about every f and every φ because the tree only ever asks it about one step.

Thank you!

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[1] L. Slavin, A. Volberg, Bellman function and the H^1–BMO duality, in: *Harmonic Analysis, Partial Differential Equations, and Related Topics*, Contemporary Mathematics, vol. 428, American Mathematical Society, 2007, pp. 113–126.

[2] D. L. Burkholder, Boundary value problems and sharp inequalities for martingale transforms, *Annals of Probability* 12 (1984), no. 3, pp. 647–702.

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[4] MATP32 Harmonic Analysis, Lectures 9 to 16, Lund University, autumn 2026.