

Harmonic Analysis

Lecture 10

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Exercise Classes: Thursdays 13.00–15.00

Recall. For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$,

$$\|f\|_{\text{BMO}(\mathbb{R}^d)} = \sup_{\substack{Q \subset \mathbb{R}^d \\ \text{cube}}} \frac{1}{|Q|} \int_Q |f(x) - \langle f \rangle_Q| dx,$$

$$\text{BMO}(\mathbb{R}^d) = \{f \in L^1_{\text{loc}}(\mathbb{R}^d) : \|f\|_{\text{BMO}(\mathbb{R}^d)} < \infty\} / \sim,$$

where $f \sim g$ if and only if $f - g$ is constant almost everywhere. *Passing to the quotient is what turns the seminorm of Lecture 9 into a norm.*

Exercise. $(\text{BMO}(\mathbb{R}^d), \|\cdot\|_{\text{BMO}(\mathbb{R}^d)})$ is a Banach space.

From Lecture 9: $L^\infty(\mathbb{R}^d) \subseteq \text{BMO}(\mathbb{R}^d)$ with $\|f\|_{\text{BMO}} \leq 2\|f\|_{L^\infty}$, and

$$f(x) = \begin{cases} \log|x| & x \neq 0, \\ 0 & x = 0, \end{cases} \quad f \in \text{BMO}(\mathbb{R}^d) \quad (\text{Exercise}).$$

Theorem (John–Nirenberg). Let $f \in \text{BMO}(\mathbb{R}^d)$ with $\|f\|_{\text{BMO}(\mathbb{R}^d)} > 0$. There are constants $C_1, C_2 > 0$ depending only on d such that for any cube Q and any $\lambda > 0$,

$$\left| \{x \in Q : |f(x) - \langle f \rangle_Q| > \lambda\} \right| \leq C_1 |Q| e^{-C_2 \lambda / \|f\|_{\text{BMO}(\mathbb{R}^d)}}.$$

Note. From the definition we immediately have the weaker bound

$$\left| \{x \in Q : |f(x) - \langle f \rangle_Q| > \lambda\} \right| \underset{\substack{\uparrow \\ \text{Chebyshev}}}{\leq} \frac{1}{\lambda} \int_Q |f(x) - \langle f \rangle_Q| dx \leq \frac{|Q| \|f\|_{\text{BMO}}}{\lambda},$$

which decays only like $1/\lambda$. The theorem upgrades this to exponential decay.

Note. The theorem really says that a BMO function cannot behave much worse than a logarithm. For $f = \log|x|$ on $Q = [-1, 1]$ the set $\{|f - \langle f \rangle_Q| > \lambda\}$ is an interval of length about $e^{-\lambda}$, which is exactly the shape of the right-hand side.

Idea of the proof. The hypothesis says that on every cube the average oscillation is at most $\|f\|_{\text{BMO}}$. The Calderón–Zygmund decomposition (Lecture 5) turns an L^1 average into a set where $|f - \langle f \rangle_Q|$ is small plus a family of cubes of small total measure. Inside each of those cubes we may start over, because BMO gives the same hypothesis on every subcube. Each round costs a fixed additive amount 2^{d+1} in the height and gains a factor $\frac{1}{2}$ in measure. After N rounds the height is about $N2^{d+1}$ and the measure about 2^{-N} , and that is exponential decay.

Proof. Fix $f \in \text{BMO}(\mathbb{R}^d)$ and a cube Q . Assume without loss of generality that $\|f\|_{\text{BMO}} = 1$ (replace f by $f/\|f\|_{\text{BMO}}$, and rescale λ at the end). Let

$$\tilde{f} := \chi_Q (f - \langle f \rangle_Q), \quad \text{so} \quad \frac{1}{|Q|} \|\tilde{f}\|_{L^1} \leq \|f\|_{\text{BMO}} = 1 \leq 2.$$

We apply the Calderón–Zygmund decomposition of Lecture 5 to $\tilde{f}$ on Q at height $a = 2$, using the dyadic subcubes $\mathcal{D}(Q)$ of Q . The hypothesis of that theorem is $\frac{1}{|Q|} \int_Q |\tilde{f}| \leq a$, which is the display above.

Case 1: $\lambda \geq 2^{d+1}$. Choose $N \in \mathbb{N}$ such that

$$N 2^{d+1} \leq \lambda < (N+1) 2^{d+1}.$$

First generation. Carrying out the Calderón–Zygmund decomposition for $\tilde{f}$ at height $a = 2$ we obtain a collection of pairwise disjoint cubes $\mathcal{Q}_1 = \{Q_{j_1}\}_{j_1 \geq 0} \subseteq \mathcal{D}(Q)$ such that

$$2 < \frac{1}{|Q_{j_1}|} \int_{Q_{j_1}} |f(x) - \langle f \rangle_Q| dx \leq 2 \cdot 2^d = 2^{d+1} \quad \text{for every } j_1, \quad (\dagger)$$

$$|f(x) - \langle f \rangle_Q| \leq 2 \quad \text{for a.e. } x \in Q \setminus \bigcup_{j_1} Q_{j_1}. \quad (3)$$

($\dagger$) and (3) are conclusions (b) and (a) of Lecture 5, Theorem 1, applied to $\tilde{f}$ with $a = 2$. Nothing about BMO is used in (3): a point x outside all stopping cubes has $\frac{1}{|Q'|} \int_{Q'} |\tilde{f}| \leq 2$ for every dyadic $Q' \ni x$, since none of them stopped, and letting Q' shrink to x the Lebesgue differentiation theorem gives $|\tilde{f}(x)| \leq 2$ a.e. The number 2 is the chosen height, not a BMO constant.

The stopping cubes are small in total measure:

$$\begin{aligned} \sum_{j_1} |Q_{j_1}| &\stackrel{\substack{\leq \\ \uparrow \\ \text{lower bound in } (\dagger)}}{\leq} \sum_{j_1} \frac{1}{2} \int_{Q_{j_1}} |f(x) - \langle f \rangle_Q| dx \stackrel{\substack{= \\ \uparrow \\ \text{cubes disjoint}}}{=} \frac{1}{2} \int_{\bigcup_{j_1} Q_{j_1}} |f(x) - \langle f \rangle_Q| dx \\ &\leq \frac{1}{2} \int_Q |f(x) - \langle f \rangle_Q| dx = \frac{1}{2} \|\tilde{f}\|_{L^1} \stackrel{\substack{\leq \\ \uparrow \\ \|f\|_{\text{BMO}} = 1}}{\leq} \frac{1}{2} |Q|. \end{aligned} \quad (4)$$

The averages do not jump much from a cube to a stopping subcube:

$$\begin{aligned} |\langle f \rangle_{Q_{j_1}} - \langle f \rangle_Q| &= \left| \frac{1}{|Q_{j_1}|} \int_{Q_{j_1}} (f(x) - \langle f \rangle_Q) dx \right| \\ &\leq \frac{1}{|Q_{j_1}|} \int_{Q_{j_1}} |f(x) - \langle f \rangle_Q| dx \stackrel{\substack{\leq \\ \uparrow \\ \text{upper bound in } (\dagger)}}{\leq} 2^{d+1}. \end{aligned} \quad (5)$$

Second generation. Now fix $Q_{j_1} \in \mathcal{Q}_1$ and let

$$\tilde{f}_{Q_{j_1}} := \chi_{Q_{j_1}} (f - \langle f \rangle_{Q_{j_1}}), \quad \frac{1}{|Q_{j_1}|} \|\tilde{f}_{Q_{j_1}}\|_{L^1} \leq \|f\|_{\text{BMO}} = 1 \leq 2.$$

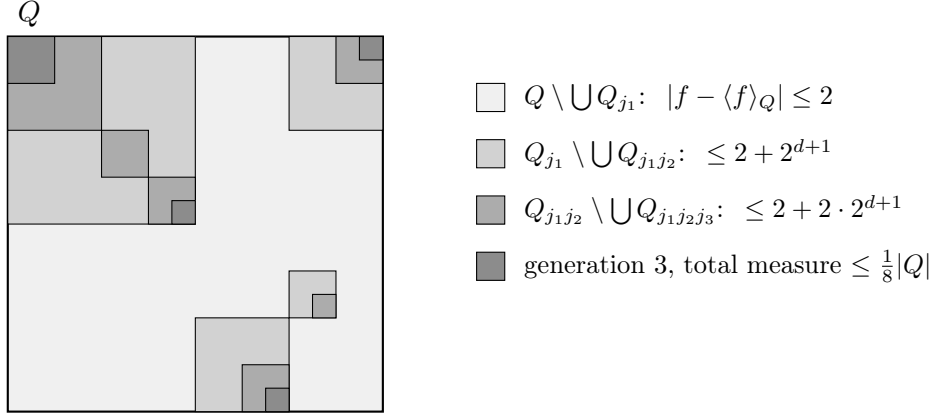
This is where BMO is used: the hypothesis of the Calderón–Zygmund decomposition holds again on Q_{j_1} , with the same height 2, because the BMO norm controls the oscillation on every cube at once. Apply the Calderón–Zygmund decomposition at height $a = 2$ again to obtain a pairwise disjoint collection

$\{Q_{j_1, j_2}\}_{j_2 \geq 0} \subseteq \mathcal{D}(Q_{j_1}) \subseteq \mathcal{D}(Q)$. We get (†), (4), (5) with Q_{j_1} in the role of Q , but (3) changes a bit: for a.e. $x \in Q_{j_1} \setminus \bigcup_{j_2} Q_{j_1, j_2}$,

$$|f(x) - \langle f \rangle_Q| \underset{\substack{\uparrow \\ \text{triangle}}}{\leq} \underbrace{|f(x) - \langle f \rangle_{Q_{j_1}}|}_{\leq 2 \text{ by (3) on } Q_{j_1}} + \underbrace{|\langle f \rangle_{Q_{j_1}} - \langle f \rangle_Q|}_{\leq 2^{d+1} \text{ by (5)}} \leq 2 + 2^{d+1} \leq 2 \cdot 2^{d+1}. \quad (3')$$

The measure halves again:

$$\sum_{j_1} \sum_{j_2} |Q_{j_1, j_2}| \underset{\substack{\uparrow \\ (4) \text{ applied to each } Q_{j_1}}}{\leq} \sum_{j_1} \frac{1}{2} |Q_{j_1}| \underset{\substack{\uparrow \\ (4) \text{ applied to } Q}}{\leq} \frac{1}{4} |Q|.$$



Three generations of the iterated Calderón-Zygmund decomposition, shown for $d = 2$. Each generation lives inside the previous one, covers at most half its measure, and raises the bound on $|f - \langle f \rangle_Q|$ by 2^{d+1} .

N generations. Do N steps and obtain a pairwise disjoint collection $\{Q_{j_1, \dots, j_N}\}_{j_1, \dots, j_N \geq 0} \subseteq \mathcal{D}(Q)$ such that for each $j_1, \dots, j_{N-1}$ and a.e. $x \in Q_{j_1 \dots j_{N-1}} \setminus \bigcup_{j_N} Q_{j_1 \dots j_N}$,

$$\begin{aligned} |f(x) - \langle f \rangle_Q| &\underset{\substack{\uparrow \\ \text{telescoping, triangle}}}{\leq} \underbrace{|f(x) - \langle f \rangle_{Q_{j_1 \dots j_{N-1}}}|}_{\leq 2 \text{ by (3)}} + \underbrace{|\langle f \rangle_{Q_{j_1 \dots j_{N-1}}} - \langle f \rangle_{Q_{j_1 \dots j_{N-2}}}|}_{\leq 2^{d+1} \text{ by (5)}} \\ &\quad + \dots + \underbrace{|\langle f \rangle_{Q_{j_1}} - \langle f \rangle_Q|}_{\leq 2^{d+1} \text{ by (5)}} \\ &\leq 2 + (N-1) 2^{d+1} \leq N 2^{d+1}. \end{aligned} \quad (5^*)$$

There are $N-1$ difference terms, one per generation. The same computation at an earlier generation $k < N$ gives the bound $2 + (k-1)2^{d+1} \leq N 2^{d+1}$, so every point of Q outside the N -th generation satisfies $|f(x) - \langle f \rangle_Q| \leq N 2^{d+1}$ almost everywhere. Iterating (4) N times,

$$\sum_{j_1} \dots \sum_{j_N} |Q_{j_1 \dots j_N}| \leq 2^{-N} |Q|. \quad (4^*)$$

Conclusion of Case 1. Since $\lambda \geq N 2^{d+1}$, (5*) gives, up to a null set,

$$\{x \in Q : |f(x) - \langle f \rangle_Q| > \lambda\} \subseteq \bigcup_{j_1, \dots, j_N} Q_{j_1 \dots j_N},$$

and the cubes are disjoint, so by (4*)

$$\left| \{x \in Q : |f(x) - \langle f \rangle_Q| > \lambda\} \right| \leq \sum_{j_1, \dots, j_N} |Q_{j_1 \dots j_N}| \leq 2^{-N} |Q| = e^{-N \log 2} |Q|.$$

From the choice of N , $N > \frac{\lambda}{2^{d+1}} - 1$, hence with

$$C_2 := \frac{\log 2}{2^{d+2}}$$

we get

$$2^{-N} < 2 \cdot 2^{-\lambda/2^{d+1}} = 2e^{-2C_2\lambda} \leq 2e^{-C_2\lambda},$$

so

$$\left| \{x \in Q : |f(x) - \langle f \rangle_Q| > \lambda\} \right| \leq 2|Q|e^{-C_2\lambda}.$$

We could take $C_2 = \log 2 / 2^{d+1}$ in Case 1; the extra factor $\frac{1}{2}$ is there to make Case 2 work with the same constants.

Case 2: $0 < \lambda < 2^{d+1}$. Here the set is at most all of Q and we only need to pay for the exponential:

$$\left| \{x \in Q : |f(x) - \langle f \rangle_Q| > \lambda\} \right| \leq |Q| = |Q|e^{C_2\lambda}e^{-C_2\lambda} \underset{C_2\lambda < \frac{\log 2}{2}}{\leq} \sqrt{2}|Q|e^{-C_2\lambda} \leq 2|Q|e^{-C_2\lambda}.$$

Hence in both cases, with $C_1 = 2$ and $C_2 = \log 2 / 2^{d+2}$, for $\|f\|_{\text{BMO}} = 1$,

$$\left| \{x \in Q : |f(x) - \langle f \rangle_Q| > \lambda\} \right| \leq C_1|Q|e^{-C_2\lambda}.$$

For general f apply this to $f/\|f\|_{\text{BMO}}$ with $\lambda/\|f\|_{\text{BMO}}$ in place of λ , which gives

$$\left| \{x \in Q : |f(x) - \langle f \rangle_Q| > \lambda\} \right| \leq C_1|Q|e^{-C_2\lambda/\|f\|_{\text{BMO}(\mathbb{R}^d)}}. \quad \blacksquare$$

Corollary. For each $p \in (1, \infty)$ there exists a constant C_p (depending on p and d) such that for each $f \in \text{BMO}(\mathbb{R}^d)$,

$$\|f\|_{\text{BMO}(\mathbb{R}^d)} \underset{(1)}{\leq} \sup_{\substack{Q \\ \text{cube}}} \left(\frac{1}{|Q|} \int_Q |f(x) - \langle f \rangle_Q|^p dx \right)^{1/p} \underset{(2)}{\leq} C_p \|f\|_{\text{BMO}(\mathbb{R}^d)}.$$

So the L^p -oscillation and the L^1 -oscillation define equivalent norms on BMO. The direction (1) was the Remark in Lecture 9, the direction (2) is where John–Nirenberg is needed.

Proof. (1): For any cube Q , by Hölder with exponents p and p' on the probability measure $dx/|Q|$,

$$\frac{1}{|Q|} \int_Q |f(x) - \langle f \rangle_Q| dx \leq \left(\frac{1}{|Q|} \int_Q |f(x) - \langle f \rangle_Q|^p dx \right)^{1/p} \underbrace{\left(\frac{1}{|Q|} \int_Q 1 dx \right)^{1/p'}}_{=1}.$$

(2): Fix a cube Q . By the layer cake formula and John–Nirenberg,

$$\begin{aligned} \frac{1}{|Q|} \int_Q |f(x) - \langle f \rangle_Q|^p dx &\underset{\substack{\uparrow \\ \text{layer cake}}}{=} \frac{p}{|Q|} \int_0^\infty \lambda^{p-1} \left| \{x \in Q : |f(x) - \langle f \rangle_Q| > \lambda\} \right| d\lambda \\ &\underset{\substack{\uparrow \\ \text{John–Nirenberg}}}{\leq} \frac{p}{|Q|} \int_0^\infty \lambda^{p-1} C_1|Q|e^{-C_2\lambda/\|f\|_{\text{BMO}}} d\lambda \\ &\underset{\substack{\uparrow \\ t = C_2\lambda/\|f\|_{\text{BMO}}}}{=} C_1 p \frac{\|f\|_{\text{BMO}}^p}{C_2^p} \int_0^\infty t^{p-1} e^{-t} dt \\ &= \frac{C_1 p \Gamma(p)}{C_2^p} \|f\|_{\text{BMO}}^p = \frac{C_1 \Gamma(p+1)}{C_2^p} \|f\|_{\text{BMO}}^p. \end{aligned}$$

Substitution: $\lambda = t\|f\|_{\text{BMO}}/C_2$, $d\lambda = (\|f\|_{\text{BMO}}/C_2) dt$, so $\lambda^{p-1}d\lambda = (\|f\|_{\text{BMO}}/C_2)^p t^{p-1}dt$. And $\Gamma(p) = \int_0^\infty t^{p-1}e^{-t} dt$, $p\Gamma(p) = \Gamma(p+1)$. Taking the p -th root and the supremum over Q ,

$$\sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - \langle f \rangle_Q|^p dx \right)^{1/p} \leq C_p \|f\|_{\text{BMO}(\mathbb{R}^d)}, \quad C_p = \frac{(C_1 \Gamma(p+1))^{1/p}}{C_2}. \quad \blacksquare$$

Remark. As $p \rightarrow \infty$, $\Gamma(p+1)^{1/p} \rightarrow \infty$ (for integers, $(n!)^{1/n} \sim n/e$ by Stirling), so $C_p \rightarrow \infty$, roughly linearly in p . This has to happen: if C_p stayed bounded as $p \rightarrow \infty$ we would get $\|f - \langle f \rangle_Q\|_{L^\infty(Q)} \leq C\|f\|_{\text{BMO}}$ for every cube, forcing $\text{BMO}(\mathbb{R}^d) = L^\infty(\mathbb{R}^d)$ modulo constants, which is false by the $\log|x|$ example. In this sense the Corollary is sharp.

Duality H^1 -BMO. Recall from Lecture 9

$$H^1(\mathbb{R}^d) = \left\{ \sum_{k=1}^{\infty} \mu_k a_{Q_k} : (\mu_k) \in \ell^1(\mathbb{N}), \{a_{Q_k}\} \text{ atoms} \right\}, \quad \|f\|_{H^1} = \inf \sum_k |\mu_k|,$$

the infimum over all such representations of f .

BMO $\subseteq (H^1)^*$. Let $b \in \text{BMO}(\mathbb{R}^d)$ and $f = \sum_k \mu_k a_{Q_k} \in H^1(\mathbb{R}^d)$. Define the pairing term by term,

$$\langle b, f \rangle := \sum_{k=1}^{\infty} \mu_k \int_{Q_k} a_{Q_k}(x) b(x) dx.$$

For $b \in L^\infty$ this is just $\int fb$, since the series for f converges in L^1 . For general $b \in \text{BMO}$ the product fb need not be integrable and the series is the definition. The estimate below shows the series converges absolutely. Then

$$\begin{aligned} |\langle b, f \rangle| &\leq \sum_{k=1}^{\infty} |\mu_k| \left| \int_{Q_k} a_{Q_k}(x) b(x) dx \right| \underset{\substack{= \\ \uparrow \\ \int_{Q_k} a_{Q_k} = 0}}{\leq} \sum_{k=1}^{\infty} |\mu_k| \left| \int_{Q_k} a_{Q_k}(x) (b(x) - \langle b \rangle_{Q_k}) dx \right| \\ &\underset{\substack{\leq \\ \uparrow \\ \text{C-S on } Q_k}}{\leq} \sum_{k=1}^{\infty} |\mu_k| \underbrace{\|a_{Q_k}\|_{L^2}}_{\leq |Q_k|^{-1/2}} \underbrace{\|\chi_{Q_k}(b - \langle b \rangle_{Q_k})\|_{L^2}}_{\leq C'_2 \|b\|_{\text{BMO}} |Q_k|^{1/2} \text{ by the Corollary, } p=2} \leq C'_2 \|b\|_{\text{BMO}} \sum_{k=1}^{\infty} |\mu_k|. \end{aligned}$$

Here C'_2 is the constant C_p of the Corollary with $p=2$, written with a prime to avoid a clash with the John-Nirenberg constant C_2 . Taking the infimum over representations of f ,

$$|\langle b, f \rangle| \leq C'_2 \|b\|_{\text{BMO}(\mathbb{R}^d)} \|f\|_{H^1(\mathbb{R}^d)} \implies \text{BMO}(\mathbb{R}^d) \subseteq (H^1(\mathbb{R}^d))^*.$$

$(H^1)^* \cap L^1_{\text{loc}} \subseteq \text{BMO}$. Now suppose that $b \in L^1_{\text{loc}}(\mathbb{R}^d)$ and that $f \mapsto \langle b, f \rangle$ is a bounded functional on $H^1(\mathbb{R}^d)$, say $|\langle b, f \rangle| \leq C\|f\|_{H^1}$. Since every atom a_Q has $\|a_Q\|_{H^1} \leq 1$,

$$\sup_{\substack{Q \\ \text{cube}}} \sup_{\substack{a_Q \\ \text{atom on } Q}} \left| \int_Q b(x) a_Q(x) dx \right| \leq C < \infty,$$

and for every atom

$$\left| \int_Q b(x) a_Q(x) dx \right| \underset{\substack{= \\ \uparrow \\ \int_Q a_Q = 0}}{\leq} \left| \int_Q (b(x) - \langle b \rangle_Q) a_Q(x) dx \right|.$$

Fix a cube Q . Any $a \in L^2(Q)$ with $\|a\|_{L^2}^2 \leq |Q|^{-1}$ can be turned into an atom by subtracting its mean: $a - \langle a \rangle_Q$ is supported on Q , has integral zero, and

$$\|a - \langle a \rangle_Q\|_{L^2(Q)} \leq \|a\|_{L^2(Q)} \leq |Q|^{-1/2},$$

$\uparrow$
subtracting the mean is an orthogonal projection in $L^2(Q)$

so $a - \langle a \rangle_Q$ is an atom on Q . Moreover subtracting the mean of a makes no difference in the pairing with $b - \langle b \rangle_Q$:

$$\int_Q (b(x) - \langle b \rangle_Q) \langle a \rangle_Q dx = \langle a \rangle_Q \int_Q (b(x) - \langle b \rangle_Q) dx = 0.$$

Hence

$$\begin{aligned} \sup_{\substack{a \in L^2(Q) \\ \|a\|_{L^2}^2 \leq |Q|^{-1}}} \left| \int_Q (b(x) - \langle b \rangle_Q) a(x) dx \right| &= \sup_{\substack{a \in L^2(Q) \\ \|a\|_{L^2}^2 \leq |Q|^{-1}}} \left| \int_Q (b(x) - \langle b \rangle_Q) (a(x) - \langle a \rangle_Q) dx \right| \\ &\leq \sup_{\substack{a - \langle a \rangle_Q \text{ is an atom} \\ \uparrow \\ \text{atom on } Q}} \left| \int_Q (b(x) - \langle b \rangle_Q) a_Q(x) dx \right| \\ &\leq C. \end{aligned}$$

The left-hand side is $|Q|^{-1/2}$ times the $L^2(Q)$ norm of $b - \langle b \rangle_Q$, by duality in the Hilbert space $L^2(Q)$. Therefore

$$\left(\frac{1}{|Q|} \int_Q |b(x) - \langle b \rangle_Q|^2 dx \right)^{1/2} \leq C \quad \text{for every cube } Q,$$

and by part (1) of the Corollary (Hölder), $\|b\|_{\text{BMO}(\mathbb{R}^d)} \leq C$. Hence $b \in \text{BMO}(\mathbb{R}^d)$. ■

Theorem (Fefferman–Stein).

$$(H^1(\mathbb{R}^d))^* = \text{BMO}(\mathbb{R}^d).$$

What was shown above: for $b \in L^1_{\text{loc}}$, the functional $f \mapsto \langle b, f \rangle$ is bounded on H^1 if and only if $b \in \text{BMO}$, with $\|b\|_{\text{BMO}}$ comparable to the operator norm. The remaining part of the theorem, that every bounded linear functional on H^1 is of the form $f \mapsto \langle b, f \rangle$ for some $b \in L^1_{\text{loc}}$, was not proved in the lecture.