

Harmonic Analysis

Lecture 11

Teodoras Paura

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Exercise Classes: Thursdays 13.00–15.00

Definition. Write

$$\mathbb{R}_+^{d+1} := \{(x, t) : x \in \mathbb{R}^d, t > 0\}$$

for the upper half-space. A positive Borel measure μ on $\mathbb{R}_+^{d+1}$ is said to be a *Carleson measure* if there exists a constant $C_\mu > 0$ such that

$$\mu(Q \times (0, \ell(Q))) \leq C_\mu |Q|$$

for every cube Q in $\mathbb{R}^d$. Here $\ell(Q)$ denotes the side length of Q . If μ is a Carleson measure, its *Carleson constant* is defined by

$$\|\mu\| := \sup_{\substack{Q \subseteq \mathbb{R}^d \\ \text{cube}}} \frac{\mu(Q \times (0, \ell(Q)))}{|Q|}.$$

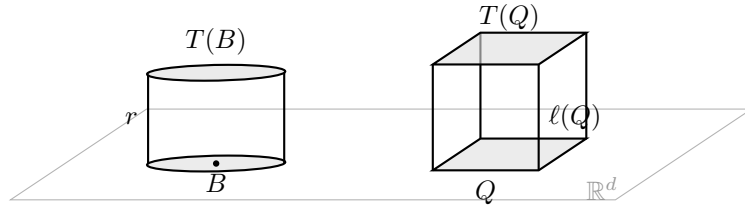
What this says: μ lives in the upper half-space, and the condition tests how much mass μ puts in the box standing on a cube Q with height equal to the side of Q . A Carleson measure is one that never puts more mass in such a box than a constant times the area of its base. The measure may be very singular, all that matters is this scale-invariant counting.

If Q is a cube in $\mathbb{R}^d$, define the *tent* over Q to be the box

$$T(Q) := Q \times (0, \ell(Q)).$$

If $B = B(x_0, r)$ is a ball in $\mathbb{R}^d$, define the *cylindrical tent* over B by

$$T(B) := \{(x, t) \in \mathbb{R}_+^{d+1} : x \in B, 0 < t \leq r\}.$$



Cylindrical tent over a ball and box tent over a cube, shown for $d = 2$. The height of each tent equals the size of its base.

We seek to prove a theorem that connects Carleson measures with certain boundedness properties of operators defined by approximate identities.

Recall: for $t > 0$ and g on $\mathbb{R}^d$ we denote

$$g_t(x) := t^{-d} g(t^{-1}x), \quad x \in \mathbb{R}^d.$$

Theorem 2. Let Φ be a non-negative integrable function on $\mathbb{R}^d$ such that

$$\Phi(x) \leq \frac{C}{(1 + |x|)^{d+\delta}} \quad \text{for a.e. } x \in \mathbb{R}^d,$$

where $C, \delta > 0$ are constants. Let μ be a positive Borel measure on $\mathbb{R}_+^{d+1}$. Then the following are equivalent:

- (i) μ is a Carleson measure.
- (ii) For every $p \in (1, \infty)$ there exists a constant $C_{p,d,\mu} > 0$ such that

$$\int_{\mathbb{R}_+^{d+1}} |(\Phi_t * f)(x)|^p d\mu(x, t) \leq C_{p,d,\mu} \int_{\mathbb{R}^d} |f(x)|^p dx.$$

For (ii) $\Rightarrow$ (i) one also needs Φ not to vanish near the origin, that is $\int_{B(0,1)} \Phi > 0$; see the proof. A non-negative approximate identity always satisfies this.

What the theorem says: the map $f \mapsto \Phi_t * f(x)$ takes a function on $\mathbb{R}^d$ to a function on $\mathbb{R}_+^{d+1}$ (its smoothed version at every scale t). The theorem characterises exactly those measures on $\mathbb{R}_+^{d+1}$ for which this map is bounded $L^p(\mathbb{R}^d) \rightarrow L^p(\mu)$, and the answer is the same for every $p \in (1, \infty)$: the Carleson condition.

For this we need some independent results.

Theorem 3 (Whitney decomposition). Let $\mathcal{O}$ be a non-empty open set in $\mathbb{R}^d$ with $\mathcal{O} \subsetneq \mathbb{R}^d$. Then there exists a countable collection $\mathcal{Q}$ of closed cubes in $\mathbb{R}^d$ such that

- (a) the cubes in $\mathcal{Q}$ have mutually disjoint interiors,
- (b) $\mathcal{O} = \bigcup_{Q \in \mathcal{Q}} Q$,
- (c) $\sqrt{d}\ell(Q) \leq \text{dist}(Q, \mathbb{R}^d \setminus \mathcal{O}) \leq 4\sqrt{d}\ell(Q)$ for all $Q \in \mathcal{Q}$.

Proof: Appendix J in [2] (Grafakos, *Classical Fourier Analysis*).

Condition (c) is the point: each Whitney cube has size comparable to its distance to the boundary of $\mathcal{O}$. Cubes near the boundary are small, cubes deep inside are large. It is the dyadic tool for decomposing an open set, the way the Calderón–Zygmund decomposition decomposes a function.

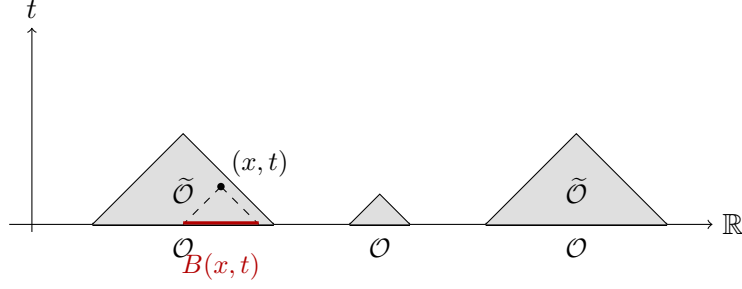
Proposition 4. Let μ be a positive Borel measure on $\mathbb{R}_+^{d+1}$. For a non-empty open set $\mathcal{O} \subsetneq \mathbb{R}^d$ write

$$\tilde{\mathcal{O}} := \{(x, t) \in \mathbb{R}_+^{d+1} : B(x, t) \subseteq \mathcal{O}\}.$$

Then the following are equivalent:

- (i) μ is a Carleson measure.
- (ii) There exists a constant $C_{\mu,d} > 0$ such that for every non-empty open set $\mathcal{O}$ in $\mathbb{R}^d$ with $\mathcal{O} \subsetneq \mathbb{R}^d$ one has

$$\mu\left(\{(x, t) \in \mathbb{R}_+^{d+1} : B(x, t) \subseteq \mathcal{O}\}\right) = \mu(\tilde{\mathcal{O}}) \leq C_{\mu,d} |\mathcal{O}|.$$



The set $\tilde{\mathcal{O}}$ for $d = 1$: $(x, t) \in \tilde{\mathcal{O}}$ means the interval $B(x, t) = (x - t, x + t)$ fits inside $\mathcal{O}$, so $\tilde{\mathcal{O}}$ is the union of the triangles standing on the components of $\mathcal{O}$. Proposition 4 says a Carleson measure gives these “tents over open sets” mass at most a constant times the length of the base, which generalises the defining condition from cubes to arbitrary open sets.

Proof of (i) $\Rightarrow$ (ii). Suppose that μ is a Carleson measure and let $\mathcal{O}$ be a non-empty open set with $\mathcal{O} \subsetneq \mathbb{R}^d$. By Theorem 3 there exists a countable collection $\mathcal{Q}$ of closed cubes in $\mathbb{R}^d$ with the properties (a), (b), (c).

Claim. If $\mathcal{O}$ and $\mathcal{Q} = \{Q\}_{Q \in \mathcal{Q}}$ are as above, then

$$\tilde{\mathcal{O}} \subseteq \bigcup_{Q \in \mathcal{Q}} Q \times (0, 11\sqrt{d}\ell(Q)).$$

Assume the Claim holds. Then

$$\begin{aligned} \mu(\tilde{\mathcal{O}}) &\leq \mu\left(\bigcup_{Q \in \mathcal{Q}} Q \times (0, 11\sqrt{d}\ell(Q))\right) \stackrel{\substack{\leq \\ \uparrow \\ \text{countable subadditivity}}}{\leq} \sum_{Q \in \mathcal{Q}} \mu\left(Q \times (0, 11\sqrt{d}\ell(Q))\right) \\ &\stackrel{\substack{\leq \\ \uparrow}}{\leq} \sum_{Q \in \mathcal{Q}} \mu\left(Q^* \times (0, \ell(Q^*))\right) \\ &\stackrel{\substack{\leq \\ \uparrow \\ \text{Carleson, definition of } \|\mu\|}}{\leq} \|\mu\| \sum_{Q \in \mathcal{Q}} |Q^*|, \end{aligned}$$

where Q^* is the dilation of Q about its centre with side length $\ell(Q^*) = 11\sqrt{d}\ell(Q)$. Since $|Q^*| = (11\sqrt{d})^d |Q| = 11^d d^{d/2} |Q|$,

$$\mu(\tilde{\mathcal{O}}) \leq 11^d d^{d/2} \|\mu\| \sum_{Q \in \mathcal{Q}} |Q| \stackrel{\substack{= \\ \uparrow \\ (a)}}{=} 11^d d^{d/2} \|\mu\| \left| \bigcup_{Q \in \mathcal{Q}} Q \right| \stackrel{\substack{= \\ \uparrow \\ (b)}}{=} 11^d d^{d/2} \|\mu\| |\mathcal{O}|.$$

Therefore (ii) holds with $C_{\mu,d} = 11^d d^{d/2} \|\mu\|$, and it remains to prove the Claim.

Proof of the Claim. Let $(x, t) \in \tilde{\mathcal{O}}$. Then $x \in B(x, t) \subseteq \mathcal{O}$, so by (b) there exists $Q \in \mathcal{Q}$ with $x \in Q$. Since Q and $\mathbb{R}^d \setminus \mathcal{O}$ are closed and Q is compact, there exist $y_0 \in \mathbb{R}^d \setminus \mathcal{O}$ and $x_0 \in Q$ such that

$$\text{dist}(Q, \mathbb{R}^d \setminus \mathcal{O}) = |x_0 - y_0|. \quad (3)$$

Consider the open ball

$$B_0 := B(y_0, 6\sqrt{d}\ell(Q)).$$

We claim

$$Q \subseteq B_0. \quad (4)$$

Indeed, if x_Q is the centre of Q , then for every $z \in Q$,

$$\begin{aligned}
|z - y_0| &\stackrel{\substack{\leq \\ \uparrow \\ \text{triangle}}}{\leq} |z - x_Q| + |x_Q - x_0| + |x_0 - y_0| \\
&\stackrel{\substack{\leq \\ \uparrow \\ z, x_0 \in Q, \text{ half-diagonal}}}{\leq} \frac{\sqrt{d}}{2}\ell(Q) + \frac{\sqrt{d}}{2}\ell(Q) + |x_0 - y_0| = \sqrt{d}\ell(Q) + |x_0 - y_0| \\
&\stackrel{\substack{\leq \\ \uparrow \\ (3) \text{ and } (c)}}{\leq} \sqrt{d}\ell(Q) + 4\sqrt{d}\ell(Q) = 5\sqrt{d}\ell(Q) < 6\sqrt{d}\ell(Q),
\end{aligned}$$

so (4) holds.

Observe that by construction $B_0 \cap (\mathbb{R}^d \setminus \mathcal{O}) \neq \emptyset$, since $y_0 \in B_0$. We show that this forces

$$t < 11\sqrt{d}\ell(Q).$$

Argue by contradiction and assume $t \geq 11\sqrt{d}\ell(Q)$. Then for every $y \in B_0$,

$$\begin{aligned}
|y - x| &\stackrel{\substack{\leq \\ \uparrow \\ \text{triangle}}}{\leq} |y - y_0| + |y_0 - x_0| + |x_0 - x| \\
&\stackrel{\substack{< \\ \uparrow \\ y \in B_0, (3) \text{ and } (c), x_0, x \in Q}}{<} 6\sqrt{d}\ell(Q) + 4\sqrt{d}\ell(Q) + \sqrt{d}\ell(Q) = 11\sqrt{d}\ell(Q) \leq t,
\end{aligned}$$

where y_0, x_0 are as in (3). Hence $y \in B(x, t)$, that is $B_0 \subseteq B(x, t) \subseteq \mathcal{O}$. But $y_0 \in B_0$ and $y_0 \notin \mathcal{O}$, a contradiction. Thus $t < 11\sqrt{d}\ell(Q)$, and since also $x \in Q$,

$$(x, t) \in Q \times (0, 11\sqrt{d}\ell(Q)) \implies \tilde{\mathcal{O}} \subseteq \bigcup_{Q \in \mathcal{Q}} Q \times (0, 11\sqrt{d}\ell(Q)). \quad \blacksquare$$

Where the numbers come from: x sits in a Whitney cube Q whose distance to the complement is at most $4\sqrt{d}\ell(Q)$ by (c). If the ball $B(x, t)$ is to stay inside $\mathcal{O}$, it cannot reach the nearest outside point y_0 , and y_0 is within $\sqrt{d}\ell(Q) + 4\sqrt{d}\ell(Q) = 5\sqrt{d}\ell(Q)$ of x . So t is bounded by a fixed multiple of $\ell(Q)$. The ball B_0 of radius $6\sqrt{d}\ell(Q)$ and the final $11\sqrt{d}$ are just comfortable choices making the inequalities strict.

Exercise. Prove (ii) $\Rightarrow$ (i) in Proposition 4.

Remark 5. Let Φ be as in the statement of Theorem 2. There exists a constant $A_{\Phi, d}$, depending on Φ and d , such that for $f \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$, one has

$$M_{\Phi}(f)(x) := \sup_{(y, t) \in \Gamma(x)} |(\Phi_t * f)(y)| \leq A_{\Phi, d} M(f)(x) \quad \text{for a.e. } x \in \mathbb{R}^d, \quad (5)$$

where M is the Hardy–Littlewood maximal operator and

$$\Gamma(x) := \{(y, t) \in \mathbb{R}_+^{d+1} : |x - y| < t\}$$

is the cone with vertex x . Compare the nontangential maximal function of Lecture 9, where Φ was the Poisson kernel. The decay hypothesis on Φ gives a radially decreasing integrable majorant, and the argument of Seminar 2, Exercise 3 then gives (5); moving from the vertical line to the cone costs only a constant because $\Phi_t(y - z) \leq C \Phi'_t(x - z)$ for $|x - y| < t$ with a slightly fatter majorant Φ' .

Proof of Theorem 2.

(i) $\Rightarrow$ (ii). Suppose that μ is a Carleson measure. Let $p \in (1, \infty)$ and let $f \in L^p(\mathbb{R}^d)$ be given. By the layer cake formula,

$$\int_{\mathbb{R}_+^{d+1}} |(\Phi_t * f)(x)|^p d\mu(x, t) = \int_0^\infty p \lambda^{p-1} \mu\left(\{(x, t) \in \mathbb{R}_+^{d+1} : |(\Phi_t * f)(x)| > \lambda\}\right) d\lambda.$$

Write

$$\mathcal{O}_\lambda := \{x \in \mathbb{R}^d : M_\Phi(f)(x) > \lambda\}, \quad \lambda > 0.$$

Exercise. For $\lambda > 0$, show that $\mathcal{O}_\lambda$ is open. (This is needed for measurability and to apply Proposition 4.)

Also $\mathcal{O}_\lambda \neq \mathbb{R}^d$: by (5), $M_\Phi f \leq A_{\Phi, d} Mf \in L^p$, so $|\mathcal{O}_\lambda| \leq \lambda^{-p} A_{\Phi, d}^p \|Mf\|_{L^p}^p < \infty$. If $\mathcal{O}_\lambda = \emptyset$ the bound below is trivial.

Key inclusion: for every $\lambda > 0$,

$$\{(x, t) \in \mathbb{R}_+^{d+1} : |(\Phi_t * f)(x)| > \lambda\} \subseteq \tilde{\mathcal{O}}_\lambda. \quad (*)$$

Indeed, let $|(\Phi_t * f)(x)| > \lambda$ and take any $y \in B(x, t)$. Then $|y - x| < t$, so $(x, t) \in \Gamma(y)$, and therefore

$$M_\Phi(f)(y) = \sup_{(z, s) \in \Gamma(y)} |(\Phi_s * f)(z)| \geq |(\Phi_t * f)(x)| > \lambda,$$

that is $y \in \mathcal{O}_\lambda$. As $y \in B(x, t)$ was arbitrary, $B(x, t) \subseteq \mathcal{O}_\lambda$, which is $(x, t) \in \tilde{\mathcal{O}}_\lambda$. The point is that the cone condition is symmetric: $(x, t) \in \Gamma(y)$ if and only if $y \in B(x, t)$. So one large value $|\Phi_t * f(x)|$ makes $M_\Phi f$ large on the whole ball $B(x, t)$, not just at x .

By the Exercise and Proposition 4 applied to $\mathcal{O}_\lambda$, together with (*), we deduce

$$\mu\left(\{(x, t) \in \mathbb{R}_+^{d+1} : |(\Phi_t * f)(x)| > \lambda\}\right) \leq \mu(\tilde{\mathcal{O}}_\lambda) \leq C_{\mu, d} |\mathcal{O}_\lambda| \quad \forall \lambda > 0,$$

where $C_{\mu, d} > 0$ is independent of f and λ . So we have

$$\begin{aligned} \int_{\mathbb{R}_+^{d+1}} |(\Phi_t * f)(x)|^p d\mu(x, t) &\leq C_{\mu, d} \int_0^\infty p \lambda^{p-1} \left| \{x \in \mathbb{R}^d : M_\Phi(f)(x) > \lambda\} \right| d\lambda \\ &\stackrel{\substack{= \\ \uparrow \\ \text{layer cake again}}}{=} C_{\mu, d} \int_{\mathbb{R}^d} (M_\Phi(f)(x))^p dx \\ &\stackrel{\substack{\leq \\ \uparrow \\ (5)}}{\leq} C_{\mu, d} A_{\Phi, d}^p \int_{\mathbb{R}^d} (M(f)(x))^p dx \\ &\stackrel{\substack{\leq \\ \uparrow \\ \text{Lecture 2 Corollary}}}{\leq} C_{\mu, d} A_{\Phi, d}^p \left(C_d \frac{p}{p-1}\right)^p \int_{\mathbb{R}^d} |f(x)|^p dx, \end{aligned}$$

using $\|M\|_{L^p \rightarrow L^p} \leq C_d \frac{p}{p-1}$. Constant from Marcinkiewicz with $C_1 = 3^d$, $C_\infty = 1$, as in Lecture 2 and Lecture 6: $\|Mf\|_{L^p}^p \leq \frac{2^p p 3^d}{p-1} \|f\|_{L^p}^p$. Hence (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). Suppose μ is a positive Borel measure on $\mathbb{R}_+^{d+1}$ such that (ii) holds. Let $B_0 := B(x_0, r)$ be an open ball in $\mathbb{R}^d$ and write $2B_0 := B(x_0, 2r)$. Let $(x, t) \in \mathbb{R}_+^{d+1}$ be such that $B(x, t) \subseteq B_0$. Then

$$|(\Phi_t * \chi_{2B_0})(x)| = \int_{2B_0} \Phi_t(x - y) dy \stackrel{\substack{= \\ \uparrow \\ y \mapsto x - y}}{=} \int_{x-2B_0} \Phi_t(y) dy.$$

Since $B(x, t) \subseteq B_0 \subseteq 2B_0$, we have $x - 2B_0 \supseteq x - B(x, t) = B(0, t)$, and so, as $\Phi_t \geq 0$,

$$|(\Phi_t * \chi_{2B_0})(x)| \geq \int_{B(0,t)} \Phi_t(y) dy \underset{\substack{\uparrow \\ y=tz}}{=} \int_{B(0,1)} \Phi(z) dz =: c_{d,\Phi} > 0.$$

This is where $\int_{B(0,1)} \Phi > 0$ is needed. The value $c_{d,\Phi}$ does not depend on t : the approximate identity always puts the same fraction of its mass in the ball of its own scale. So on the set $E := \{(x, t) \in \mathbb{R}_+^{d+1} : B(x, t) \subseteq B_0\}$ we have $c_{d,\Phi}^p \chi_E \leq |\Phi_t * \chi_{2B_0}|^p$ pointwise, and integrating against μ gives $c_{d,\Phi}^p \mu(E) \leq \int_{\mathbb{R}_+^{d+1}} |\Phi_t * \chi_{2B_0}|^p d\mu$. Hence

$$\begin{aligned} \mu\left(\{(x, t) \in \mathbb{R}_+^{d+1} : B(x, t) \subseteq B_0\}\right) &\underset{\substack{\uparrow \\ \text{Chebyshev, see above}}}{\leq} \frac{1}{c_{d,\Phi}^p} \int_{\mathbb{R}_+^{d+1}} |(\Phi_t * \chi_{2B_0})(x)|^p d\mu(x, t) \\ &\underset{\substack{\uparrow \\ (ii) \text{ with } f = \chi_{2B_0}}}{\leq} \frac{C_{p,d,\mu}}{c_{d,\Phi}^p} \int_{\mathbb{R}^d} |\chi_{2B_0}(x)|^p dx \\ &= \frac{C_{p,d,\mu}}{c_{d,\Phi}^p} |2B_0| = \frac{2^d C_{p,d,\mu}}{c_{d,\Phi}^p} |B_0|. \end{aligned}$$

This is condition (ii) of Proposition 4 for balls, and by (ii) $\Rightarrow$ (i) of Proposition 4 (the Exercise) μ is a Carleson measure. ■

Summary of the proof. (i) $\Rightarrow$ (ii) is the real content: the level sets of $\Phi_t * f$ in the half-space sit inside tents over the level sets of the nontangential maximal function on $\mathbb{R}^d$, Proposition 4 converts μ -mass of tents into Lebesgue measure of the base, and then everything reduces to the Hardy–Littlewood maximal theorem. (ii) $\Rightarrow$ (i) is a test on indicator functions of balls: the convolution with the approximate identity sees a definite fraction $c_{d,\Phi}$ of the ball from every point of the tent above it.

Corollary 6. Let μ be a positive Borel measure on $\mathbb{R}_+^{d+1}$. The following are equivalent:

- (i) μ is a Carleson measure.
- (ii) For every $p \in (1, \infty)$ there exists a constant $C_{p,d,\mu} > 0$ such that

$$\left(\int_{\mathbb{R}_+^{d+1}} |(P_t * f)(x)|^p d\mu(x, t) \right)^{1/p} \leq C_{p,d,\mu} \|f\|_{L^p(\mathbb{R}^d)}.$$

Here $(P_t)_{t>0}$ is the Poisson kernel on $\mathbb{R}^d$,

$$P_t(x) := \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \frac{t}{(t^2 + |x|^2)^{\frac{d+1}{2}}}, \quad x \in \mathbb{R}^d, t > 0.$$

This is Theorem 2 with $\Phi = P_1$, which is non-negative, integrable, strictly positive, and satisfies $P_1(x) \leq C(1+|x|)^{-d-1}$, so the hypotheses hold with $\delta = 1$, and $P_t = (P_1)_t$ in the dilation notation above. In words: a measure on the upper half-space is Carleson exactly when the Poisson extension $u(x, t) = P_t * f(x)$ of every L^p boundary function is in $L^p(\mu)$. This is Carleson's original theorem, proved for the corona problem.