

Harmonic Analysis

Lecture 12

Teodoras Paura

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Exercise Classes: Thursdays 13.00–15.00

We return to the theorem of Lecture 11 and give the proof once more, this time running the maximal function estimate through the radially decreasing majorant of Φ rather than through the nontangential maximal function.

Theorem 1. Let $\Phi : \mathbb{R}^d \rightarrow [0, \infty)$ be measurable with

$$\Phi(x) \leq \frac{C}{(1 + |x|)^{d+\delta}} =: \tilde{\Phi}(x) \quad \text{for a.e. } x \in \mathbb{R}^d,$$

where $C, \delta > 0$ are constants. Let μ be a positive Borel measure on $\mathbb{R}_+^{d+1}$. Then the following are equivalent:

- (i) μ is a Carleson measure.
- (ii) For every $p \in (1, \infty)$ there exists a constant $C_{p,\mu,d,\Phi} > 0$ such that

$$\int_{\mathbb{R}_+^{d+1}} |(\Phi_t * f)(x)|^p d\mu(x, t) \leq C_{p,\mu,d,\Phi} \|f\|_{L^p(\mathbb{R}^d)}^p \quad \forall f \in L^p(\mathbb{R}^d).$$

For (ii) $\Rightarrow$ (i) one needs in addition $c_{d,\Phi} := \int_{B(0,1)} \Phi > 0$, otherwise $\Phi = 0$ satisfies (ii) for every μ . Every non-negative approximate identity has this property.

Two facts are used. First, from the exercises: if $\Psi \geq 0$ is radial, $\Psi(x) = \Psi_0(|x|)$ with Ψ_0 non-increasing and having only finitely many discontinuities, then

$$|(\Psi * f)(x)| \leq \|\Psi\|_{L^1} M_c f(x) \quad \text{for all } x \in \mathbb{R}^d, \quad (*)$$

where M_c is the centred Hardy–Littlewood maximal function. The dilates Ψ_t have the same properties and $\|\Psi_t\|_{L^1} = \|\Psi\|_{L^1}$, so $(*)$ holds for every Ψ_t with the same constant. The majorant Φ is radial, decreasing, continuous and integrable, with

$$C_{\tilde{\Phi}} := \|\tilde{\Phi}\|_{L^1} = C \int_{\mathbb{R}^d} \frac{dx}{(1 + |x|)^{d+\delta}} < \infty,$$

so $(*)$ applies to $\tilde{\Phi}_t$. Second, Proposition 4 of Lecture 11: μ is Carleson if and only if

$$\mu(\tilde{\mathcal{O}}) \leq C_{\mu,d} |\mathcal{O}| \quad \text{for every open } \mathcal{O} \subsetneq \mathbb{R}^d, \quad \tilde{\mathcal{O}} := \{(x, t) \in \mathbb{R}_+^{d+1} : B(x, t) \subseteq \mathcal{O}\}.$$

Proof of (i) $\Rightarrow$ (ii). Let μ be Carleson, $p \in (1, \infty)$ and $f \in L^p(\mathbb{R}^d)$. Since $0 \leq \Phi \leq \tilde{\Phi}$,

$$|(\Phi_t * f)(x)| \leq (\tilde{\Phi}_t * |f|)(x) \quad \text{for all } (x, t) \in \mathbb{R}_+^{d+1},$$

so it suffices to bound the right-hand side. By the layer cake formula,

$$\int_{\mathbb{R}_+^{d+1}} (\tilde{\Phi}_t * |f|)(x)^p d\mu(x, t) = \int_0^\infty p \lambda^{p-1} \mu(E_\lambda) d\lambda, \quad E_\lambda := \{(x, t) \in \mathbb{R}_+^{d+1} : (\tilde{\Phi}_t * |f|)(x) > \lambda\}.$$

Define, with M the uncentred Hardy–Littlewood maximal function,

$$\mathcal{O}_\lambda := \{y \in \mathbb{R}^d : 2^{d+\delta} C_{\tilde{\Phi}} Mf(y) > \lambda\}, \quad \lambda > 0.$$

$\mathcal{O}_\lambda$ is open, since Mf is lower semicontinuous: if $\frac{1}{|B|} \int_B |f| > a$ for some open ball $B \ni y$, the same ball witnesses $Mf > a$ at every point of B . Also $\mathcal{O}_\lambda \neq \mathbb{R}^d$, because $|\mathcal{O}_\lambda| \leq \lambda^{-p} (2^{d+\delta} C_{\tilde{\Phi}})^p \|Mf\|_{L^p}^p < \infty$.

Key inclusion. For every $\lambda > 0$,

$$E_\lambda \subseteq \tilde{\mathcal{O}}_\lambda. \quad (1)$$

Let $(x, t) \in E_\lambda$ and $y \in B(x, t)$. For every $z \in \mathbb{R}^d$ we have $|y - z| \leq |y - x| + |x - z| < t + |x - z|$, hence

$$1 + \frac{|y - z|}{t} < 2 + \frac{|x - z|}{t} \leq 2 \left(1 + \frac{|x - z|}{t}\right),$$

and therefore

$$\tilde{\Phi}_t(x - z) = \frac{C t^{-d}}{(1 + \frac{|x - z|}{t})^{d+\delta}} \leq \frac{2^{d+\delta} C t^{-d}}{(1 + \frac{|y - z|}{t})^{d+\delta}} = 2^{d+\delta} \tilde{\Phi}_t(y - z).$$

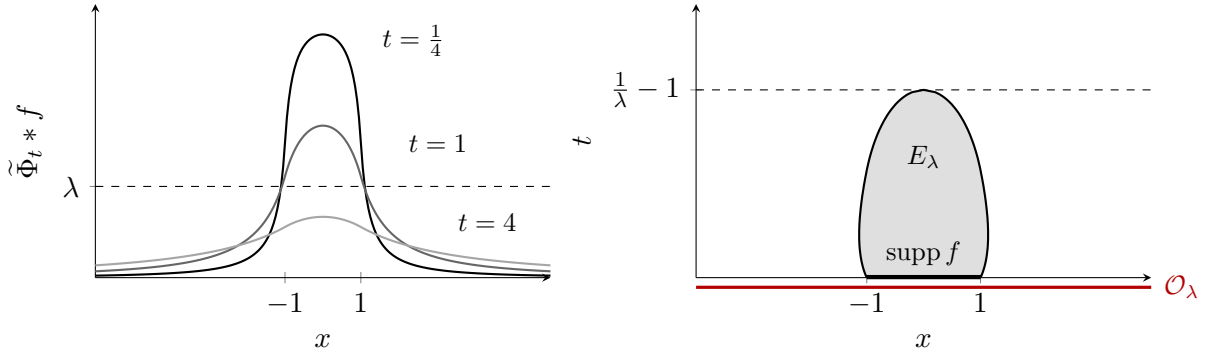
Multiplying by $|f(z)|$ and integrating in z ,

$$\lambda < (\tilde{\Phi}_t * |f|)(x) \leq 2^{d+\delta} (\tilde{\Phi}_t * |f|)(y) \underset{(*)}{\leq} 2^{d+\delta} C_{\tilde{\Phi}} M_c f(y) \leq 2^{d+\delta} C_{\tilde{\Phi}} Mf(y),$$

that is $y \in \mathcal{O}_\lambda$. As $y \in B(x, t)$ was arbitrary, $B(x, t) \subseteq \mathcal{O}_\lambda$, which is (1).

*Moving the centre of $\tilde{\Phi}_t$ by less than t changes it by at most the factor $2^{d+\delta}$. This is what turns a large value of $\tilde{\Phi}_t * |f|$ at the single point x into a large value of Mf on the whole ball $B(x, t)$, and that is exactly what membership in $\tilde{\mathcal{O}}_\lambda$ requires. Without the factor one only gets $x \in \mathcal{O}_\lambda$, which says nothing about μ .*

*The set E_λ is bounded in t : by Hölder with the conjugate exponent q , $(\tilde{\Phi}_t * |f|)(x) \leq \|\tilde{\Phi}_t\|_{L^q} \|f\|_{L^p} = t^{-d/p} \|\tilde{\Phi}\|_{L^q} \|f\|_{L^p}$, which tends to 0 as $t \rightarrow \infty$. Convolution at large scale spreads the mass of f out, so large t never contributes to E_λ .*



*Left: $d = 1$, $\delta = 1$, $\tilde{\Phi}(x) = \frac{1}{2}(1 + |x|)^{-2}$ (mass one), $f = \chi_{[-1,1]}$. The convolution $\tilde{\Phi}_t * f$ has total mass $\|f\|_{L^1} = 2$ for every t , so as t grows it flattens and its maximum $\tilde{\Phi}_t * f(0) = \frac{1}{1+t}$ drops below λ .*

*Right: the level set $E_\lambda = \{(x, t) \in \mathbb{R}_+^2 : \tilde{\Phi}_t * f(x) > \lambda\}$ in the half-plane. It is bounded in t by $\frac{1}{\lambda} - 1$ and has width about the support of f . Picking t large is never an option: large t means the mass of f has been spread too thin to exceed λ anywhere. In red on the axis is the set $\mathcal{O}_\lambda \subseteq \mathbb{R}$ from the proof: here $Mf(y) = \min\{1, \frac{2}{1+|y|}\}$ and $2^{d+\delta} C_{\tilde{\Phi}} = 4$, so $\mathcal{O}_\lambda = \{|y| < \frac{8}{\lambda} - 1\}$, an interval of half-length about 25.7 for $\lambda = 0.3$. The tent $\tilde{\mathcal{O}}_\lambda$ over it is the triangle of that height, far larger than the window shown, so (1) holds with lots of room. The inclusion is crude, it only needs to be true.*

Now (1), Proposition 4 of Lecture 11 applied to the open set $\mathcal{O}_\lambda$, and the layer cake formula on $\mathbb{R}^d$ give

$$\begin{aligned} \int_{\mathbb{R}_+^{d+1}} (\tilde{\Phi}_t * |f|)(x)^p d\mu(x, t) &\stackrel{(1)}{\leq} \int_0^\infty p \lambda^{p-1} \mu(\tilde{\mathcal{O}}_\lambda) d\lambda \stackrel{\substack{\uparrow \\ \text{Proposition 4}}}{\leq} C_{\mu,d} \int_0^\infty p \lambda^{p-1} |\mathcal{O}_\lambda| d\lambda \\ &\stackrel{\substack{\uparrow \\ \text{layer cake}}}{=} C_{\mu,d} \int_{\mathbb{R}^d} (2^{d+\delta} C_{\tilde{\Phi}} Mf(x))^p dx \\ &\stackrel{\substack{\uparrow \\ \text{Lecture 2 Corollary}}}{\leq} C_{\mu,d} (2^{d+\delta} C_{\tilde{\Phi}})^p \left(C_d \frac{p}{p-1}\right)^p \|f\|_{L^p(\mathbb{R}^d)}^p. \end{aligned}$$

This is (ii) with $C_{p,\mu,d,\Phi} = C_{\mu,d} (2^{d+\delta} C_{\tilde{\Phi}} C_d \frac{p}{p-1})^p$. ■

Proof of (ii) $\Rightarrow$ (i). Assume (ii) and $c_{d,\Phi} = \int_{B(0,1)} \Phi > 0$.

Step 1: balls. Let $B_0 = B(x_0, r)$ be any open ball and $2B_0 = B(x_0, 2r)$. For $(x, t) \in \mathbb{R}_+^{d+1}$ with $B(x, t) \subseteq B_0$,

$$\begin{aligned} (\Phi_t * \chi_{2B_0})(x) &= \int_{2B_0} \Phi_t(x-y) dy \stackrel{\substack{\uparrow \\ y \mapsto x-y}}{=} \int_{x-2B_0} \Phi_t(y) dy \\ &\stackrel{\substack{\uparrow \\ B(0,t) = x - B(x,t) \subseteq x - 2B_0}}{\geq} \int_{B(0,t)} \Phi_t(y) dy \stackrel{\substack{\uparrow \\ y=tz}}{=} \int_{B(0,1)} \Phi(z) dz = c_{d,\Phi}. \end{aligned}$$

So $c_{d,\Phi}^p \chi_{\widetilde{B_0}} \leq (\Phi_t * \chi_{2B_0})^p$ pointwise on $\mathbb{R}_+^{d+1}$, where $\widetilde{B_0} = \{(x, t) : B(x, t) \subseteq B_0\}$, and integrating against μ ,

$$\mu(\widetilde{B_0}) \leq \frac{1}{c_{d,\Phi}^p} \int_{\mathbb{R}_+^{d+1}} |(\Phi_t * \chi_{2B_0})(x)|^p d\mu(x, t) \stackrel{\substack{\uparrow \\ (ii), f = \chi_{2B_0}}}{\leq} \frac{C_{p,\mu,d,\Phi}}{c_{d,\Phi}^p} |2B_0| = \frac{2^d C_{p,\mu,d,\Phi}}{c_{d,\Phi}^p} |B_0|. \quad (2)$$

Step 2: from balls to cubes. Let Q be a cube with centre c_Q and let $B_Q := B(c_Q, (1 + \frac{\sqrt{d}}{2})\ell(Q))$. If $(x, t) \in T(Q) = Q \times (0, \ell(Q))$ and $z \in B(x, t)$, then

$$|z - c_Q| \leq |z - x| + |x - c_Q| < t + \frac{\sqrt{d}}{2}\ell(Q) < (1 + \frac{\sqrt{d}}{2})\ell(Q),$$

so $B(x, t) \subseteq B_Q$. Hence $T(Q) \subseteq \widetilde{B_Q}$, and by (2)

$$\mu(T(Q)) \leq \mu(\widetilde{B_Q}) \leq \frac{2^d C_{p,\mu,d,\Phi}}{c_{d,\Phi}^p} |B_Q| = \frac{2^d C_{p,\mu,d,\Phi} v_d (1 + \frac{\sqrt{d}}{2})^d}{c_{d,\Phi}^p} |Q|,$$

with $v_d = |B(0, 1)|$. This is the Carleson condition. ■

Step 2 is the reason it is enough to test Carleson-type conditions on balls, cubes or cylinders: a tent over a cube sits inside the “cone tent” $\widetilde{B}$ over a ball of comparable size. The same two lines prove (ii) $\Rightarrow$ (i) of Proposition 4 in Lecture 11 (take $\mathcal{O} = B_Q$).

Corollary 2. Let μ be a positive Borel measure on $\mathbb{R}_+^{d+1}$. Then μ is a Carleson measure if and only if for every $p \in (1, \infty)$ there exists $C_{p,d,\mu} > 0$ with

$$\int_{\mathbb{R}_+^{d+1}} |(P_t * f)(x)|^p d\mu(x, t) \leq C_{p,d,\mu} \|f\|_{L^p(\mathbb{R}^d)}^p \quad \forall f \in L^p(\mathbb{R}^d),$$

where $P_t = (P)_t$ is the Poisson kernel,

$$P(x) = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \frac{1}{(1 + |x|^2)^{\frac{d+1}{2}}}, \quad P_t(x) = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \frac{t}{(t^2 + |x|^2)^{\frac{d+1}{2}}}.$$

Proof: Theorem 1 with $\Phi = P$, which is positive and satisfies $P(x) \leq C(1 + |x|)^{-d-1}$, so $\delta = 1$. ■

*This is the kernel used all the time: $u(x, t) = P_t * f(x)$ is the harmonic extension of f to the upper half-space, and the Corollary says the Carleson measures are exactly the measures on $\mathbb{R}_+^{d+1}$ against which harmonic extensions of L^p functions are L^p -integrable.*

Carleson measures and BMO

Theorem 3. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \psi(x) dx = 0$. Then for every $b \in \text{BMO}(\mathbb{R}^d)$ the measure μ on $\mathbb{R}_+^{d+1}$ given by

$$d\mu(x, t) = |(b * \psi_t)(x)|^2 dx \frac{dt}{t}$$

is a Carleson measure, and

$$\|\mu\| \leq C_{d,\psi} \|b\|_{\text{BMO}}^2,$$

where $C_{d,\psi} > 0$ depends only on ψ and d .

What this says: $b * \psi_t(x)$ measures the part of b living at scale t near x (the mean zero condition kills constants, so only oscillation is seen). For $b \in L^2$ the total mass $\int_{\mathbb{R}_+^{d+1}} |b * \psi_t|^2 dx \frac{dt}{t}$ is comparable to $\|b\|_{L^2}^2$ by Plancherel. For $b \in \text{BMO}$ the total mass is infinite in general, but the mass over any tent is controlled by the area of its base, and the constant is the square of the BMO norm. Together with the Corollary this is the bridge between BMO on $\mathbb{R}^d$ and the half-space.

The converse is also true: if ψ is in addition non-degenerate (for instance $\hat{\psi} \neq 0$ on every ray from the origin) and the measure above is Carleson, then $b \in \text{BMO}$. More generally the balayage $S\mu(x) := \int_{\mathbb{R}_+^{d+1}} P_t(x-y) d\mu(y, t)$ of any Carleson measure μ is a BMO function with $\|S\mu\|_{\text{BMO}} \leq C_d \|\mu\|$ (Carleson, Fefferman–Stein). So the correspondence goes in both directions. We will not prove the converse in the course.

Before proving Theorem 3 we need to know that $b * \psi_t$ makes sense for $b \in \text{BMO}$, which is only locally integrable and can grow like $\log|x|$. The two lemmas below handle this: averages of a BMO function over nested cubes grow at most linearly in the number of doublings, and that growth is beaten by any polynomial decay.

Lemma 4. Let $b \in \text{BMO}(\mathbb{R}^d)$ and let $Q \subseteq \mathbb{R}^d$ be a cube. Then for each $N \in \mathbb{N}$,

$$|\langle b \rangle_Q - \langle b \rangle_{2^N Q}| \leq 2^d N \|b\|_{\text{BMO}(\mathbb{R}^d)}.$$

Here $2^N Q$ is the cube with the same centre as Q and side length $2^N \ell(Q)$.

Proof. It suffices to show that for every cube Q

$$|\langle b \rangle_Q - \langle b \rangle_{2Q}| \leq 2^d \|b\|_{\text{BMO}}. \quad (3)$$

Once (3) is known we apply it to $Q, 2Q, \dots, 2^{N-1}Q$ and telescope:

$$|\langle b \rangle_Q - \langle b \rangle_{2^N Q}| \leq \sum_{k=0}^{N-1} |\langle b \rangle_{2^k Q} - \langle b \rangle_{2^{k+1} Q}| \underset{(3) \text{ for } 2^k Q}{\leq} N 2^d \|b\|_{\text{BMO}}.$$

For (3), since $\langle b \rangle_Q - \langle b \rangle_{2Q} = \frac{1}{|Q|} \int_Q (b(x) - \langle b \rangle_{2Q}) dx$,

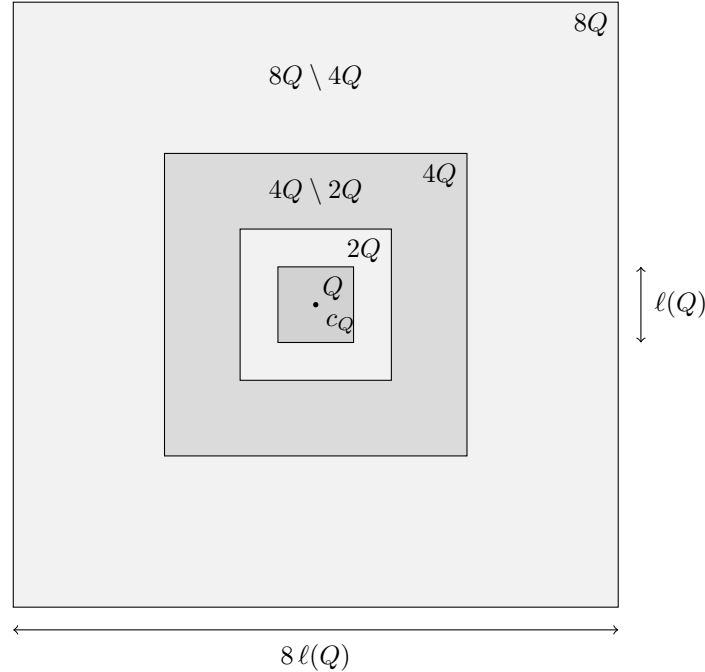
$$\begin{aligned} |\langle b \rangle_Q - \langle b \rangle_{2Q}| &\leq \frac{1}{|Q|} \int_Q |b(x) - \langle b \rangle_{2Q}| dx \\ &\underset{Q \subseteq 2Q, |2Q| = 2^d |Q|}{\leq} 2^d \frac{1}{|2Q|} \int_{2Q} |b(x) - \langle b \rangle_{2Q}| dx \leq 2^d \|b\|_{\text{BMO}}. \quad \blacksquare \end{aligned}$$

Each doubling of the cube can move the average by at most $2^d \|b\|_{\text{BMO}}$, so after N doublings the average has moved by at most $2^d N \|b\|_{\text{BMO}}$. For $b = \log|x|$ and Q centred at the origin, doubling the cube changes the average by exactly $\log 2$, so the linear growth in N is sharp.

Lemma 5. Let $\delta > 0$. There exists $C_{d,\delta} > 0$ such that for every $b \in \text{BMO}(\mathbb{R}^d)$ and every cube $Q \subseteq \mathbb{R}^d$ with centre c_Q ,

$$\ell(Q)^\delta \int_{\mathbb{R}^d} \frac{|b(x) - \langle b \rangle_Q|}{(\ell(Q) + |x - c_Q|)^{d+\delta}} dx \leq C_{d,\delta} \|b\|_{\text{BMO}}.$$

What this says: the weight $\ell(Q)^\delta (\ell(Q) + |x - c_Q|)^{-d-\delta} = \ell(Q)^{-d} (1 + \frac{|x - c_Q|}{\ell(Q)})^{-d-\delta}$ is the dilate $\tilde{\Phi}_{\ell(Q)}(x - c_Q)$ of the majorant from Theorem 1 (with $C = 1$). So the lemma bounds the oscillation of b around its average on Q , measured against a bump of scale $\ell(Q)$ centred at Q that sees all of $\mathbb{R}^d$ with polynomial decay, uniformly in Q . By Lemma 4 the oscillation of b at distance $2^N \ell(Q)$ from Q is of size $N \|b\|_{\text{BMO}}$, and the weight is of size $2^{-N\delta}$ there, so the sum converges. In particular $b * \psi_t(x)$ is absolutely convergent for $b \in \text{BMO}$, $\psi \in \mathcal{S}$ with mean zero: write $b * \psi_t(x) = \int (b(y) - \langle b \rangle_Q) \psi_t(x - y) dy$ with Q the cube of side t centred at x , and use $|\psi_t(x - y)| \leq C \tilde{\Phi}_t(x - y)$.



The plane is chopped into Q and the dyadic annuli $2^{N+1}Q \setminus 2^N Q$, $N \geq 0$ (shown for $d = 2$). On the N -th annulus $|x - c_Q| \geq 2^{N-1} \ell(Q)$, so the weight is at most $2^{-(N-1)(d+\delta)} \ell(Q)^{-d}$, while the area is $2^{(N+1)d} |Q|$ and the oscillation around $\langle b \rangle_Q$ is controlled by $\|b\|_{\text{BMO}}$ plus the average shift from Lemma 4.

Proof. Reduction to $Q_0 = [-\frac{1}{2}, \frac{1}{2}]^d$. Let $Q = c_Q + \ell(Q)Q_0$ and put $b_Q(y) := b(c_Q + \ell(Q)y)$. Translations and dilations map cubes to cubes and preserve averages, so $\langle b_Q \rangle_{Q_0} = \langle b \rangle_Q$ and $\|b_Q\|_{\text{BMO}} = \|b\|_{\text{BMO}}$. Substituting $x = c_Q + \ell(Q)y$, $dx = \ell(Q)^d dy$,

$$\begin{aligned} \ell(Q)^\delta \int_{\mathbb{R}^d} \frac{|b(x) - \langle b \rangle_Q|}{(\ell(Q) + |x - c_Q|)^{d+\delta}} dx &= \ell(Q)^\delta \int_{\mathbb{R}^d} \frac{|b_Q(y) - \langle b_Q \rangle_{Q_0}|}{\ell(Q)^{d+\delta} (1 + |y|)^{d+\delta}} \ell(Q)^d dy \\ &= \int_{\mathbb{R}^d} \frac{|b_Q(y) - \langle b_Q \rangle_{Q_0}|}{(1 + |y|)^{d+\delta}} dy. \end{aligned}$$

So it suffices to prove the lemma for Q_0 , with $\ell(Q_0) = 1$, $c_{Q_0} = 0$, $|Q_0| = 1$, and b in place of b_Q .

The cube Q_0 . Split $\mathbb{R}^d = Q_0 \cup \bigcup_{N=0}^{\infty} (2^{N+1}Q_0 \setminus 2^N Q_0)$, a disjoint union. On Q_0 the weight is at

most 1, so

$$\int_{Q_0} \frac{|b(x) - \langle b \rangle_{Q_0}|}{(1 + |x|)^{d+\delta}} dx \leq \int_{Q_0} |b(x) - \langle b \rangle_{Q_0}| dx \underset{|Q_0|=1}{\leq} \|b\|_{\text{BMO}}.$$

If $x \in 2^{N+1}Q_0 \setminus 2^N Q_0$ then $x \notin [-2^{N-1}, 2^{N-1}]^d$, so some coordinate satisfies $|x_i| > 2^{N-1}$ and $1 + |x| > 2^{N-1}$. Hence, by the triangle inequality $|b(x) - \langle b \rangle_{Q_0}| \leq |b(x) - \langle b \rangle_{2^{N+1}Q_0}| + |\langle b \rangle_{2^{N+1}Q_0} - \langle b \rangle_{Q_0}|$,

$$\begin{aligned} & \int_{2^{N+1}Q_0 \setminus 2^N Q_0} \frac{|b(x) - \langle b \rangle_{Q_0}|}{(1 + |x|)^{d+\delta}} dx \\ & \leq \frac{1}{2^{(N-1)(d+\delta)}} \left[\int_{2^{N+1}Q_0} |b(x) - \langle b \rangle_{2^{N+1}Q_0}| dx + \int_{2^{N+1}Q_0} |\langle b \rangle_{2^{N+1}Q_0} - \langle b \rangle_{Q_0}| dx \right] \\ & \underset{\substack{\leq \\ \uparrow \\ \text{BMO, Lemma 4 with } N+1}}{\leq} \frac{1}{2^{(N-1)(d+\delta)}} \left[|2^{N+1}Q_0| \|b\|_{\text{BMO}} + |2^{N+1}Q_0| (N+1) 2^d \|b\|_{\text{BMO}} \right] \\ & \underset{\substack{= \\ \uparrow \\ |2^{N+1}Q_0| = 2^{(N+1)d}}}{=} 2^{2d+\delta} 2^{-N\delta} (1 + 2^d (N+1)) \|b\|_{\text{BMO}}. \end{aligned}$$

Summing over $N \geq 0$ and using $\sum_{N \geq 0} 2^{-N\delta} = \frac{1}{1-2^{-\delta}}$ and $\sum_{N \geq 0} (N+1) 2^{-N\delta} = \frac{1}{(1-2^{-\delta})^2}$,

$$\int_{\mathbb{R}^d} \frac{|b(x) - \langle b \rangle_{Q_0}|}{(1 + |x|)^{d+\delta}} dx \leq \left(1 + 2^{2d+\delta} \left[\frac{1}{1-2^{-\delta}} + \frac{2^d}{(1-2^{-\delta})^2} \right] \right) \|b\|_{\text{BMO}} =: C_{d,\delta} \|b\|_{\text{BMO}}. \quad \blacksquare$$

The hypothesis $\delta > 0$ is used exactly once, to make the geometric series converge. For $\delta = 0$ the weight $(1 + |x|)^{-d}$ is not integrable and the lemma fails already for $b = \log |x|$.