

Harmonic Analysis

Lecture 13

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Exercise Classes: Thursdays 13.00–15.00

Recap: the key inclusion. Let $\Phi \geq 0$ with $\Phi(x) \leq C(1 + |x|)^{-d-\delta}$, $\delta > 0$, so that $\int_{\mathbb{R}^d} \Phi < \infty$. With the cone $\Gamma(x) = \{(y, t) \in \mathbb{R}_+^{d+1} : |x - y| < t\}$ and the nontangential maximal function

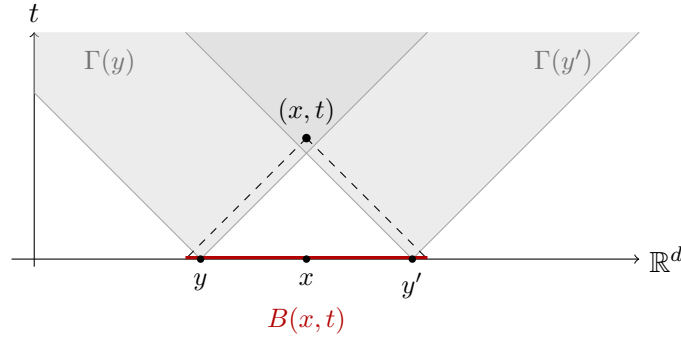
$$M_\Phi f(x) := \sup_{(y,t) \in \Gamma(x)} |(\Phi_t * f)(y)|, \quad \mathcal{O}_\lambda := \{x \in \mathbb{R}^d : M_\Phi f(x) > \lambda\},$$

$$\tilde{\mathcal{O}}_\lambda := \{(x, t) \in \mathbb{R}_+^{d+1} : B(x, t) \subseteq \mathcal{O}_\lambda\},$$

the claim used in the proof of Theorem 2 of Lecture 11 was

$$\{(x, t) \in \mathbb{R}_+^{d+1} : |(\Phi_t * f)(x)| > \lambda\} \subseteq \tilde{\mathcal{O}}_\lambda.$$

Suppose $|(\Phi_t * f)(x)| > \lambda$ and let $y \in B(x, t)$. Then $|x - y| < t$, that is $(x, t) \in \Gamma(y)$, so $M_\Phi f(y) \geq |(\Phi_t * f)(x)| > \lambda$. Hence $B(x, t) \subseteq \{y : M_\Phi f(y) > \lambda\} = \mathcal{O}_\lambda$, which is $(x, t) \in \tilde{\mathcal{O}}_\lambda$.



For $d = 1$: the cones $\Gamma(y)$ with vertex $y \in B(x, t)$ all contain the point (x, t) , because $(x, t) \in \Gamma(y)$ is the same condition as $y \in B(x, t)$. So a large value of $\Phi_t * f$ at (x, t) makes $M_\Phi f$ large at every y in the ball, and the ball sits in $\mathcal{O}_\lambda$.

Theorem 1. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \psi(x) dx = \hat{\psi}(0) = 0$. For $b \in \text{BMO}(\mathbb{R}^d)$ let

$$d\mu(x, t) := |(b * \psi_t)(x)|^2 dx \frac{dt}{t} \quad \text{on } \mathbb{R}_+^{d+1}.$$

Then μ is a Carleson measure and

$$\|\mu\| \leq C_{d,\psi} \|b\|_{\text{BMO}}^2 \quad \text{for all } b \in \text{BMO}(\mathbb{R}^d).$$

We use the two lemmas of Lecture 12. Lemma 4: for every cube Q and $N \in \mathbb{N}$, $|\langle b \rangle_Q - \langle b \rangle_{2^N Q}| \leq 2^d N \|b\|_{\text{BMO}}$. Lemma 5: for every $\delta > 0$ there is $C_{d,\delta}$ such that for every cube Q with centre c_Q ,

$$\ell(Q)^\delta \int_{\mathbb{R}^d} \frac{|b(y) - \langle b \rangle_Q|}{(\ell(Q) + |y - c_Q|)^{d+\delta}} dy \leq C_{d,\delta} \|b\|_{\text{BMO}}.$$

We also use the John–Nirenberg Corollary of Lecture 10 with $p = 2$: $(\frac{1}{|Q|} \int_Q |b - \langle b \rangle_Q|^2)^{1/2} \leq C_2 \|b\|_{\text{BMO}}$ for every cube Q .

Proof. Let Q be a cube in $\mathbb{R}^d$ with centre c_Q , and let Q^* be the cube with the same centre and $\ell(Q^*) = 3\sqrt{d}\ell(Q)$. We must show $\mu(T(Q)) \leq C_{d,\psi} \|b\|_{\text{BMO}}^2 |Q|$, where $T(Q) = Q \times (0, \ell(Q))$. By definition of μ ,

$$\mu(T(Q)) = \int_{\mathbb{R}_+^{d+1}} \chi_{Q \times (0, \ell(Q))}(x, t) d\mu(x, t) = \int_Q \int_0^{\ell(Q)} |(b * \psi_t)(x)|^2 \frac{dt}{t} dx.$$

Split b into a local part, a far part and a constant:

$$b = \underbrace{(b - \langle b \rangle_Q) \chi_{Q^*}}_{=: b_1} + \underbrace{(b - \langle b \rangle_Q) \chi_{\mathbb{R}^d \setminus Q^*}}_{=: b_2} + \langle b \rangle_Q.$$

Since $\int \psi_t = 0$, the constant disappears: $\langle b \rangle_Q * \psi_t = \langle b \rangle_Q \int_{\mathbb{R}^d} \psi_t = 0$. So $b * \psi_t = b_1 * \psi_t + b_2 * \psi_t$, and with $|u + v|^2 \leq 2|u|^2 + 2|v|^2$,

$$\mu(T(Q)) \leq \underbrace{2 \int_Q \int_0^{\ell(Q)} |(b_1 * \psi_t)(x)|^2 \frac{dt}{t} dx}_{=: A} + \underbrace{2 \int_Q \int_0^{\ell(Q)} |(b_2 * \psi_t)(x)|^2 \frac{dt}{t} dx}_{=: B}.$$

Both convolutions converge absolutely: $b_1 \in L^2$ has compact support, and $b_2 * \psi_t$ is handled by Lemma 5 below. The mean zero condition is used twice, here to kill the constant and again inside A .

The local part A . Enlarge the domain of integration to $\mathbb{R}^d \times (0, \infty)$ and use Tonelli:

$$\begin{aligned} A &\leq \int_0^\infty \left(\int_{\mathbb{R}^d} |(b_1 * \psi_t)(x)|^2 dx \right) \frac{dt}{t} \\ &\quad \stackrel{\substack{= \\ \uparrow \\ \text{Plancherel, } \widehat{\psi_t}(\xi) = \widehat{\psi}(t\xi)}}{=} \int_0^\infty \int_{\mathbb{R}^d} |\widehat{b_1}(\xi)|^2 |\widehat{\psi}(t\xi)|^2 d\xi \frac{dt}{t} \\ &\stackrel{\substack{= \\ \uparrow \\ \text{Tonelli}}}{=} \int_{\mathbb{R}^d} |\widehat{b_1}(\xi)|^2 \left(\int_0^\infty |\widehat{\psi}(t\xi)|^2 \frac{dt}{t} \right) d\xi. \end{aligned}$$

Exercise. Show that $|\widehat{\psi}(\eta)| \leq C_\psi \min\{|\eta|, |\eta|^{-1}\}$ for all $\eta \in \mathbb{R}^d$. Near 0 use $\widehat{\psi}(0) = 0$ and the mean value theorem, $|\widehat{\psi}(\eta)| \leq \|\nabla \widehat{\psi}\|_\infty |\eta|$; at infinity use that $\widehat{\psi}$ is Schwartz.

The inner t -integral is bounded independently of $\xi \neq 0$: splitting at $t = 1/|\xi|$,

$$\int_0^\infty |\widehat{\psi}(t\xi)|^2 \frac{dt}{t} \leq C_\psi^2 \left(\int_0^{1/|\xi|} t^2 |\xi|^2 \frac{dt}{t} + \int_{1/|\xi|}^\infty \frac{1}{t^2 |\xi|^2} \frac{dt}{t} \right) = C_\psi^2 \left(\frac{1}{2} + \frac{1}{2} \right) = C_\psi^2.$$

The substitution $s = t|\xi|$ shows the integral does not depend on $|\xi|$ at all, the measure $\frac{dt}{t}$ is dilation invariant. The mean zero condition is what makes the $t \rightarrow 0$ end converge.

Hence, by Plancherel once more,

$$A \leq C_\psi^2 \int_{\mathbb{R}^d} |\widehat{b_1}(\xi)|^2 d\xi = C_\psi^2 \int_{\mathbb{R}^d} |b_1(x)|^2 dx = C_\psi^2 \int_{Q^*} |b(x) - \langle b \rangle_Q|^2 dx.$$

To bound the last integral, insert $\langle b \rangle_{Q^*}$:

$$\begin{aligned} \int_{Q^*} |b - \langle b \rangle_Q|^2 dx &\leq 2 \int_{Q^*} |b - \langle b \rangle_{Q^*}|^2 dx + 2 |Q^*| |\langle b \rangle_{Q^*} - \langle b \rangle_Q|^2 \\ &\stackrel{\substack{\leq \\ \uparrow \\ \text{John–Nirenberg } p=2}}{\leq} 2 C_2^2 |Q^*| \|b\|_{\text{BMO}}^2 + 2 |Q^*| |\langle b \rangle_{Q^*} - \langle b \rangle_Q|^2, \end{aligned}$$

and since $Q \subseteq Q^*$,

$$|\langle b \rangle_Q - \langle b \rangle_{Q^*}| \leq \frac{1}{|Q|} \int_Q |b - \langle b \rangle_{Q^*}| dx \leq \frac{|Q^*|}{|Q|} \frac{1}{|Q^*|} \int_{Q^*} |b - \langle b \rangle_{Q^*}| dx \leq (3\sqrt{d})^d \|b\|_{\text{BMO}}.$$

With $|Q^*| = (3\sqrt{d})^d |Q|$ this gives

$$A \leq 2C_\psi^2 (3\sqrt{d})^d \left(C_2^2 + (3\sqrt{d})^{2d} \right) \|b\|_{\text{BMO}}^2 |Q|.$$

The far part B . Here the Fourier transform is useless (nothing is in L^2) and we estimate point-wise. Since $\psi \in \mathcal{S}$ there is C'_ψ with $|\psi(z)| \leq C'_\psi (1 + |z|)^{-d-1}$. Let $x \in Q$, $0 < t < \ell(Q)$ and $y \notin Q^*$. Then $\|y - c_Q\|_\infty \geq \frac{3\sqrt{d}}{2} \ell(Q)$ and $\|x - c_Q\|_\infty \leq \frac{1}{2} \ell(Q)$, so

$$|x - y| \geq \|x - y\|_\infty \geq \frac{3\sqrt{d} - 1}{2} \ell(Q) \geq \ell(Q) > t,$$

and also $\ell(Q) + |y - c_Q| \leq \ell(Q) + |y - x| + |x - c_Q| \leq (2 + \frac{\sqrt{d}}{2}) |x - y|$. Therefore

$$\begin{aligned} |\psi_t(x - y)| &= t^{-d} \left| \psi\left(\frac{x - y}{t}\right) \right| \leq C'_\psi t^{-d} \left(\frac{|x - y|}{t} \right)^{-d-1} = C'_\psi \frac{t}{|x - y|^{d+1}} \\ &\leq C'_\psi \left(2 + \frac{\sqrt{d}}{2}\right)^{d+1} \frac{t}{(\ell(Q) + |y - c_Q|)^{d+1}}. \end{aligned}$$

Integrating against $|b_2(y)| = |b(y) - \langle b \rangle_Q| \chi_{\mathbb{R}^d \setminus Q^*}(y)$ and applying Lemma 5 with $\delta = 1$,

$$\begin{aligned} |(b_2 * \psi_t)(x)| &\leq C'_\psi \left(2 + \frac{\sqrt{d}}{2}\right)^{d+1} \frac{t}{\ell(Q)} \ell(Q) \int_{\mathbb{R}^d} \frac{|b(y) - \langle b \rangle_Q|}{(\ell(Q) + |y - c_Q|)^{d+1}} dy \\ &\stackrel{\substack{\leq \\ \uparrow \\ \text{Lemma 5}}}{\leq} C'_\psi \left(2 + \frac{\sqrt{d}}{2}\right)^{d+1} C_{d,1} \|b\|_{\text{BMO}} \frac{t}{\ell(Q)}. \end{aligned}$$

Squaring and integrating over $T(Q)$ against $\frac{dt}{t} dx$,

$$\begin{aligned} B &\leq \left(C'_\psi \left(2 + \frac{\sqrt{d}}{2}\right)^{d+1} C_{d,1} \right)^2 \|b\|_{\text{BMO}}^2 \int_Q \int_0^{\ell(Q)} \frac{t^2}{\ell(Q)^2} \frac{dt}{t} dx \\ &= \frac{1}{2} \left(C'_\psi \left(2 + \frac{\sqrt{d}}{2}\right)^{d+1} C_{d,1} \right)^2 \|b\|_{\text{BMO}}^2 |Q|. \end{aligned}$$

Adding the two bounds, $\mu(T(Q)) \leq 2A + 2B \leq C_{d,\psi} \|b\|_{\text{BMO}}^2 |Q|$ for every cube Q , which is the Carleson condition with $\|\mu\| \leq C_{d,\psi} \|b\|_{\text{BMO}}^2$. ■

Structure of the proof. The tent $T(Q)$ only sees scales $t < \ell(Q)$ near Q . The part of b living on Q^* is an L^2 function, and Plancherel plus the dilation invariance of $\frac{dt}{t}$ turn its contribution into $\|b_1\|_{L^2}^2$, which John–Nirenberg bounds by $|Q| \|b\|_{\text{BMO}}^2$. The part of b outside Q^* is seen only through the tail of ψ_t , which at distance $\geq \ell(Q)$ and scale $t < \ell(Q)$ is of size $t |x - y|^{-d-1}$. That extra factor t is what makes $\int_0^{\ell(Q)} t^2 \frac{dt}{t}$ finite, and Lemma 5 says the weighted tail of b is bounded by $\|b\|_{\text{BMO}}$. The decomposition only works because $\int \psi = 0$ lets us subtract $\langle b \rangle_Q$ for free.

Balayage. For a positive Borel measure μ on $\mathbb{R}_+^{d+1}$ define its *balayage* (sweeping) by

$$\mathcal{B}(\mu)(x) := \int_{\mathbb{R}_+^{d+1}} P_t(x - y) d\mu(y, t), \quad x \in \mathbb{R}^d,$$

where P_t is the Poisson kernel.

Theorem 2. If μ is a Carleson measure then $\mathcal{B}(\mu) \in \text{BMO}(\mathbb{R}^d)$ and $\|\mathcal{B}(\mu)\|_{\text{BMO}} \leq C_d \|\mu\|$.

Not proved in the course. What this says: each unit of μ -mass sitting at height t above y is replaced by a Poisson bump on $\mathbb{R}^d$ of width t centred at y , so the measure in the half-space is swept down onto the boundary. Theorem 1 goes up (a BMO function on $\mathbb{R}^d$ produces a Carleson measure in $\mathbb{R}_+^{d+1}$), Theorem 2 comes back down, and together they say Carleson measures and BMO are two descriptions of the same objects.

Dyadic BMO

Recall the dyadic cubes in $\mathbb{R}^d$,

$$\mathcal{D} := \left\{ [2^{-k}m_1, 2^{-k}(m_1+1)) \times \cdots \times [2^{-k}m_d, 2^{-k}(m_d+1)) : k \in \mathbb{Z}, m_1, \dots, m_d \in \mathbb{Z} \right\}.$$

Definition. For $b \in L^1_{\text{loc}}(\mathbb{R}^d)$,

$$\|b\|_{\text{BMO}_{\mathcal{D}}(\mathbb{R}^d)} := \sup_{Q \in \mathcal{D}} \frac{1}{|Q|} \int_Q |b(x) - \langle b \rangle_Q| dx,$$

and $\text{BMO}_{\mathcal{D}}(\mathbb{R}^d)$ is the space of b with $\|b\|_{\text{BMO}_{\mathcal{D}}} < \infty$ (modulo constants).

Since the supremum runs over fewer cubes, $\|b\|_{\text{BMO}_{\mathcal{D}}} \leq \|b\|_{\text{BMO}}$, so $\text{BMO}(\mathbb{R}^d) \subseteq \text{BMO}_{\mathcal{D}}(\mathbb{R}^d)$, and the inclusion is strict:

$$\text{BMO}_{\mathcal{D}}(\mathbb{R}^d) \supsetneq \text{BMO}(\mathbb{R}^d).$$

Example for $d = 1$: $b(x) = \log \frac{1}{x} \chi_{(0,1)}(x)$. Dyadic intervals never straddle 0, so on each of them b is either 0 or a piece of $\log \frac{1}{x}$ and the dyadic oscillation is bounded. But on $(-\varepsilon, \varepsilon)$ half the interval sees 0 and the other half sees values of size $\log \frac{1}{\varepsilon}$, so the oscillation is unbounded as $\varepsilon \rightarrow 0$ and $b \notin \text{BMO}$. Dyadic BMO cannot see across the boundaries of dyadic cubes.

The John–Nirenberg inequality holds for $\text{BMO}_{\mathcal{D}}$ with the same proof: the Calderón–Zygmund decomposition in Lecture 10 was dyadic from the start, so nothing in the argument used non-dyadic cubes.

Programme for $H^1(\mathbb{R}^d)$. Atoms, Hilbert and Riesz transforms, duality with BMO, and next the square function characterisation, for which we need conditional expectations and martingales.

Conditional expectation and martingales

- If $Q, Q' \in \mathcal{D}$ then either $Q \cap Q' = \emptyset$, or $Q \subseteq Q'$, or $Q' \subseteq Q$.
- Let $\mathcal{D}_k := \{Q \in \mathcal{D} : \ell(Q) = 2^{-k}\}$, $k \in \mathbb{Z}$. For each k , $\mathbb{R}^d = \bigcup_{Q \in \mathcal{D}_k} Q$ is a disjoint union.
- For each $x \in \mathbb{R}^d$ and each $k \in \mathbb{Z}$ there is a unique $Q_{k,x} \in \mathcal{D}_k$ with $x \in Q_{k,x}$, and

$$\{x\} = \bigcap_{k \in \mathbb{N}} Q_{k,x}, \quad Q_{k+1,x} \subseteq Q_{k,x}.$$

Let $\sigma(\mathcal{D}_k)$ be the σ -algebra generated by $\mathcal{D}_k$,

$$\sigma(\mathcal{D}_k) = \left\{ \bigcup_{Q \in \mathcal{D}_k} Q : E \subseteq \mathcal{D}_k \right\}.$$

Every cube in $\mathcal{D}_{k-1}$ is the union of its 2^d children in $\mathcal{D}_k$, so $\mathcal{D}_{k-1} \subseteq \sigma(\mathcal{D}_k)$, hence

$$\sigma(\mathcal{D}_{k-1}) \subseteq \sigma(\mathcal{D}_k) \quad (k \in \mathbb{Z}),$$

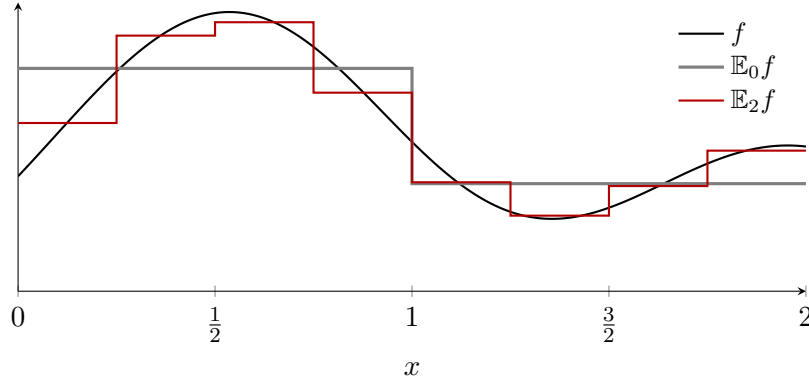
an increasing family of σ -algebras, the *dyadic filtration*.

Definition. Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. The *conditional expectation* of f with respect to $\sigma(\mathcal{D}_k)$ is

$$\mathbb{E}_k[f](x) := \sum_{Q \in \mathcal{D}_k} \langle f \rangle_Q \chi_Q(x), \quad k \in \mathbb{Z},$$

and the *martingale differences* are $\mathbb{D}_k[f] := \mathbb{E}_{k+1}[f] - \mathbb{E}_k[f]$.

What this is: $\mathbb{E}_k f$ replaces f on every dyadic cube of side 2^{-k} by its average there. It is the best $\sigma(\mathcal{D}_k)$ -measurable approximation of f , in the sense that for $f \in L^2$ it is the orthogonal projection onto the functions constant on each $Q \in \mathcal{D}_k$. The difference $\mathbb{D}_k f$ lives at scale 2^{-k} exactly: on each $Q \in \mathcal{D}_k$ it is constant on the 2^d children and has mean zero over Q . It is the dyadic analogue of $b * \psi_t$ at $t = 2^{-k}$ in Theorem 1, and the sequence $(\mathbb{E}_k f)_k$ is a martingale with respect to the dyadic filtration, $\mathbb{E}_j[\mathbb{E}_k f] = \mathbb{E}_j f$ for $j \leq k$.



$d = 1$: $\mathbb{E}_0 f$ averages f over the dyadic intervals of length 1, $\mathbb{E}_2 f$ over those of length $\frac{1}{4}$. Refining the scale, $\mathbb{E}_k f \rightarrow f$. The difference $\mathbb{D}_k f = \mathbb{E}_{k+1} f - \mathbb{E}_k f$ is the staircase one gets by splitting each step into two.

Theorem 3. For $f \in L^2(\mathbb{R}^d)$:

- (a) $f = \sum_{k \in \mathbb{Z}} \mathbb{D}_k f$, both in L^2 and almost everywhere, where $\sum_{k \in \mathbb{Z}}$ means $\lim_{N \rightarrow \infty} \sum_{k=-N}^{N-1}$.
- (b) $\mathbb{E}_k f \rightarrow f$ as $k \rightarrow \infty$, both in L^2 and almost everywhere.
- (c) (“Parseval”) $\|f\|_{L^2}^2 = \sum_{k \in \mathbb{Z}} \|\mathbb{D}_k f\|_{L^2}^2$.

Not proved today. Where it comes from: (b) almost everywhere is Lebesgue differentiation along the dyadic cubes $Q_{k,x}$, exactly Question 2(c) of Assignment 1, and in L^2 it is the convergence of orthogonal projections onto an increasing family of subspaces whose union is dense. The sum in (a) telescopes, $\sum_{k=-N}^{N-1} \mathbb{D}_k f = \mathbb{E}_N f - \mathbb{E}_{-N} f$, so (a) is (b) together with $\mathbb{E}_{-N} f \rightarrow 0$ as $N \rightarrow \infty$, which holds because $|\langle f \rangle_Q| \leq |Q|^{-1/2} \|f\|_{L^2} \rightarrow 0$ on cubes of side 2^N . (c) is orthogonality: $\mathbb{D}_j f \perp \mathbb{D}_k f$ for $j \neq k$, since $\mathbb{D}_k f$ has mean zero on every cube of $\mathcal{D}_k$ while $\mathbb{D}_j f$ for $j < k$ is constant on those cubes. This is the dyadic version of the identity $\int_{\mathbb{R}^{d+1}_+} |f * \psi_t|^2 dx \frac{dt}{t} \simeq \|f\|_{L^2}^2$ behind Theorem 1.