

Harmonic Analysis

Lecture 14

Teodoras Paura

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Exercise Classes: Thursdays 13.00–15.00

Course information.

- No lecture on Tuesday (project work). Lecture on Friday and next Tuesday are the last lectures.
- Seminar time is a workshop for the project.
- 30/10: hand in draft to Sandra and to the project partner.
- 6/11: project partner submits feedback, about half a page, not with an LLM.
- 13/11: submit final version.
- 20 minute oral presentations, roughly 25/11. Blackboard talk plus slides: use slides for definitions and formulas, develop the ideas on the blackboard.

Recap. The dyadic cubes are

$$\mathcal{D} = \left\{ [m_1 2^{-k}, (m_1 + 1) 2^{-k}) \times \cdots \times [m_d 2^{-k}, (m_d + 1) 2^{-k}) : m_1, \dots, m_d, k \in \mathbb{Z} \right\},$$

$$\mathcal{D}_k = \{Q \in \mathcal{D} : \ell(Q) = 2^{-k}\}, \quad \sigma(\mathcal{D}_k) = \left\{ \bigcup_{Q \in E} Q : E \subseteq \mathcal{D}_k \right\},$$

and $\mathcal{D}_k$ is a partition of $\mathbb{R}^d$ for each $k \in \mathbb{Z}$. Let $x \in \mathbb{R}^d$. For each $k \in \mathbb{Z}$ there exists a unique $Q_k(x) \in \mathcal{D}_k$ with $x \in Q_k(x)$, and $\{x\} = \bigcap_{k \in \mathbb{Z}} Q_k(x)$. For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ write

$$\mathbb{E}_k f = \sum_{Q \in \mathcal{D}_k} \chi_Q \langle f \rangle_Q \quad (\text{conditional expectation with respect to } \sigma(\mathcal{D}_k)),$$

$$\mathbb{D}_k f = \mathbb{E}_k f - \mathbb{E}_{k-1} f.$$

Convention: in this lecture $\mathbb{D}_k = \mathbb{E}_k - \mathbb{E}_{k-1}$, so $\mathbb{D}_k f$ lives on the cubes of $\mathcal{D}_{k-1}$ and is constant on their children in $\mathcal{D}_k$. Lecture 13 wrote $\mathbb{E}_{k+1} - \mathbb{E}_k$ for the same object with the index shifted by one.

Conditional expectation via Radon–Nikodym

We use the Radon–Nikodym theorem to define conditional expectation in general.

Radon–Nikodym Theorem. Let $(X, \mathcal{A})$ be a measurable space and let ν, μ be σ -finite measures on $(X, \mathcal{A})$ with $\nu \ll \mu$, that is $\mu(A) = 0 \Rightarrow \nu(A) = 0$ for every $A \in \mathcal{A}$. Then there exists an $\mathcal{A}$ -measurable $f : X \rightarrow [0, \infty]$ with

$$\nu(A) = \int_A f d\mu \quad \text{for all } A \in \mathcal{A},$$

and f is unique μ -a.e.

Definition of conditional expectation. Let X be a set and let $\mathcal{A}, \mathcal{B}$ be σ -algebras on X with $\mathcal{A} \subseteq \mathcal{B}$. Let μ be a measure on $(X, \mathcal{B})$ which is σ -finite already on $\mathcal{A}$, that is $X = \bigcup_n A_n$ with

$A_n \in \mathcal{A}$ and $\mu(A_n) < \infty$. Let $f : X \rightarrow [0, \infty)$ be $\mathcal{B}$ -measurable with $\int_{A_n} f d\mu < \infty$ for all n . Then there exists an $\mathcal{A}$ -measurable $g : X \rightarrow [0, \infty)$ such that

$$\int_A f d\mu = \int_A g d\mu \quad \text{for all } A \in \mathcal{A}.$$

This is exactly a consequence of Radon–Nikodym: μ is also a measure on $(X, \mathcal{A})$, and

$$\nu(A) := \int_A f d\mu, \quad A \in \mathcal{A},$$

is a measure on $(X, \mathcal{A})$ (countable additivity is monotone convergence), it is σ -finite because $\nu(A_n) < \infty$, and $\nu \ll \mu$ because $\mu(A) = 0$ forces $\int_A f d\mu = 0$. Radon–Nikodym gives an $\mathcal{A}$ -measurable $g \geq 0$ with

$$\int_A g d\mu = \nu(A) = \int_A f d\mu \quad \text{for all } A \in \mathcal{A},$$

and $g < \infty$ a.e. since $\int_{A_n} g d\mu < \infty$. We write

$$g = \mathbb{E}(f \mid \mathcal{A}),$$

unique up to μ -null sets. For real-valued f apply this to f^+ and f^- and subtract.

Do we need f integrable, and why is ν σ -finite? Neither is automatic, and the two questions are the same question. The theorem needs both μ and ν to be σ -finite on $\mathcal{A}$, and σ -finiteness of μ on the bigger σ -algebra $\mathcal{B}$ does not give it on $\mathcal{A}$: for $X = \mathbb{R}$, μ Lebesgue measure, $\mathcal{A} = \{\emptyset, \mathbb{R}\}$, the only $\mathcal{A}$ -sets have measure 0 or ∞ . There $\mathbb{E}(f \mid \mathcal{A})$ would have to be a constant c with $\int_{\mathbb{R}} c dx = \int_{\mathbb{R}} f dx$, impossible for $f \in L^1$ with $\int f \neq 0$. So the right hypothesis is the one above: X is exhausted by $\mathcal{A}$ -sets on which f is integrable. If $f \in L^1(\mu)$ this holds trivially, with ν finite. In our dyadic setting it holds for every $f \in L^1_{\text{loc}}$, because the cubes of $\mathcal{D}_k$ exhaust $\mathbb{R}^d$, lie in $\sigma(\mathcal{D}_k)$, and $\int_Q |f| < \infty$ for each of them.

Back to $\mathbb{E}_k$. Take $X = \mathbb{R}^d$, $\mathcal{B}$ the Lebesgue σ -algebra, $\mathcal{A} = \sigma(\mathcal{D}_k)$, μ Lebesgue measure, $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. We check that $\mathbb{E}_k f = \mathbb{E}(f \mid \sigma(\mathcal{D}_k))$, that is, $\mathbb{E}_k f$ has the two defining properties.

Measurability. A function g is $\sigma(\mathcal{D}_k)$ -measurable if and only if g is constant on each $Q \in \mathcal{D}_k$: if g were not constant on some Q , the inverse image of a suitable set would be a proper subset of Q , which is not a union of cubes of $\mathcal{D}_k$ and so not in $\sigma(\mathcal{D}_k)$. Conversely a function constant on each cube has inverse images that are unions of cubes. $\mathbb{E}_k f$ is constant on each $Q \in \mathcal{D}_k$ by definition, so it is $\sigma(\mathcal{D}_k)$ -measurable.

The right integrals. For $Q \in \mathcal{D}_k$,

$$\int_Q \mathbb{E}_k f dx = \int_Q \chi_Q \langle f \rangle_Q dx = |Q| \langle f \rangle_Q = \int_Q f dx,$$

and every $A \in \sigma(\mathcal{D}_k)$ is a countable disjoint union of cubes of $\mathcal{D}_k$, so by countable additivity

$$\int_A \mathbb{E}_k f dx = \int_A f dx \quad \text{for all } A \in \sigma(\mathcal{D}_k).$$

This is exactly what we want from a conditional expectation: measurable with respect to the right σ -algebra, and giving the right integrals on that σ -algebra. Hence

$$\mathbb{E}_k f = \mathbb{E}(f \mid \sigma(\mathcal{D}_k)).$$

Example with a probability measure μ and a single event A with $0 < \mu(A) < 1$. Then $\sigma(A) = \{\emptyset, X, A, A^c\}$ and

$$\mathbb{E}(f \mid \sigma(A)) = \begin{cases} \frac{1}{\mu(A)} \int_A f d\mu & \text{on } A, \\ \frac{1}{\mu(A^c)} \int_{A^c} f d\mu & \text{on } A^c, \end{cases}$$

the average of f over whichever of A, A^c one is in. $\mathbb{E}_k f$ is the same thing with the partition $\{A, A^c\}$ replaced by the partition $\mathcal{D}_k$.

The dyadic Plancherel theorem

Theorem 1. Let $f \in L^2(\mathbb{R}^d)$. Then

(a) $f = \sum_{k=-\infty}^{\infty} \mathbb{D}_k f = \lim_{k \rightarrow \infty} \mathbb{E}_k f$, with convergence in L^2 and almost everywhere, where $\sum_{k=-\infty}^{\infty}$ means $\lim_{N, M \rightarrow \infty} \sum_{k=-N}^M$;

(b) (“Plancherel”) $\|f\|_{L^2(\mathbb{R}^d)}^2 = \sum_{k=-\infty}^{\infty} \|\mathbb{D}_k f\|_{L^2(\mathbb{R}^d)}^2$.

Proof of (a). Let $f \in L^2(\mathbb{R}^d) \subseteq L^1_{\text{loc}}(\mathbb{R}^d)$. By the generalised Lebesgue differentiation theorem (Assignment 1, Question 2(c)),

$$f(x) = \lim_{k \rightarrow \infty} \langle f \rangle_{Q_k(x)} = \lim_{k \rightarrow \infty} \mathbb{E}_k f(x) \quad \text{for a.e. } x \in \mathbb{R}^d.$$

In the other direction, by Cauchy–Schwarz, for every x and k ,

$$\begin{aligned} |\mathbb{E}_k f(x)| &= \frac{1}{|Q_k(x)|} \left| \int_{Q_k(x)} f(y) dy \right| \leq \frac{1}{2^{-kd}} |Q_k(x)|^{1/2} \left(\int_{\mathbb{R}^d} |f(y)|^2 dy \right)^{1/2} \\ &= 2^{kd/2} \|f\|_{L^2} \xrightarrow{k \rightarrow -\infty} 0. \end{aligned}$$

A cube of side 2^{-k} with $k \rightarrow -\infty$ is huge, and the average of an L^2 function over a huge cube is small. Since the sum telescopes,

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \mathbb{D}_k f &= \lim_{N, M \rightarrow \infty} \sum_{k=-N}^M \mathbb{D}_k f = \lim_{N, M \rightarrow \infty} \sum_{k=-N}^M (\mathbb{E}_k f - \mathbb{E}_{k-1} f) \\ &= \lim_{N, M \rightarrow \infty} (\mathbb{E}_M f - \mathbb{E}_{-N-1} f) = f - 0 = f \quad \text{a.e.} \end{aligned}$$

Hence $\sum_{k=-\infty}^{\infty} \mathbb{D}_k f = \lim_{k \rightarrow \infty} \mathbb{E}_k f = f$ almost everywhere.

Now the L^2 convergence. We use dominated convergence. Let M' be the uncentred Hardy–Littlewood maximal function over cubes, $M'f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f|$. Since $Q_k(x)$ is a cube containing x , $|\mathbb{E}_k f(x)| \leq M'f(x)$ for every x and k , and $|f(x)| \leq M'f(x)$ for a.e. x by Lebesgue differentiation. So

$$|\mathbb{E}_k f(x) - f(x)| \leq 2M'f(x) \quad \text{for a.e. } x \in \mathbb{R}^d,$$

and the right-hand side is independent of k . M' over cubes is comparable to M over balls, a cube containing x sits inside a ball containing x of comparable measure and conversely, so M' is bounded on L^2 by the Lecture 2 Corollary. Thus $(2M'f)^2 \in L^1(\mathbb{R}^d)$ with $\|(2M'f)^2\|_{L^1} = 4\|M'f\|_{L^2}^2 \leq C_d \|f\|_{L^2}^2$, the integrands $|\mathbb{E}_k f - f|^2 \rightarrow 0$ a.e. as $k \rightarrow \infty$, and dominated convergence gives

$$\int_{\mathbb{R}^d} |\mathbb{E}_k f(x) - f(x)|^2 dx \xrightarrow{k \rightarrow \infty} 0.$$

Similarly $|\mathbb{E}_k f|^2 \leq (M' f)^2$ and $\mathbb{E}_k f \rightarrow 0$ pointwise as $k \rightarrow -\infty$, so by the same argument

$$\int_{\mathbb{R}^d} |\mathbb{E}_k f(x)|^2 dx \xrightarrow{k \rightarrow -\infty} 0.$$

Therefore $\sum_{k=-N}^M \mathbb{D}_k f = \mathbb{E}_M f - \mathbb{E}_{-N-1} f \rightarrow f$ in L^2 as $N, M \rightarrow \infty$, and (a) is done.

Proof of (b). First we write $\mathbb{D}_k f$ explicitly. For $R \in \mathcal{D}_{k-1}$ the cubes $Q \in \mathcal{D}_k$ with $Q \subsetneq R$ are the 2^d children of R , and

$$\begin{aligned} \mathbb{D}_k f &= \mathbb{E}_k f - \mathbb{E}_{k-1} f = \sum_{Q \in \mathcal{D}_k} \chi_Q \langle f \rangle_Q - \sum_{R \in \mathcal{D}_{k-1}} \chi_R \langle f \rangle_R \\ &= \sum_{R \in \mathcal{D}_{k-1}} \left(\sum_{\substack{Q \in \mathcal{D}_k \\ Q \subsetneq R}} \chi_Q \langle f \rangle_Q - \chi_R \langle f \rangle_R \right). \end{aligned}$$

Note that, with $|R| = 2^d |Q|$ for each child Q ,

$$\langle f \rangle_R = \frac{1}{|R|} \sum_{\substack{Q \in \mathcal{D}_k \\ Q \subsetneq R}} \int_Q f(x) dx = 2^{-d} \sum_{\substack{Q \in \mathcal{D}_k \\ Q \subsetneq R}} \frac{1}{|Q|} \int_Q f(x) dx = 2^{-d} \sum_{\substack{Q \in \mathcal{D}_k \\ Q \subsetneq R}} \langle f \rangle_Q.$$

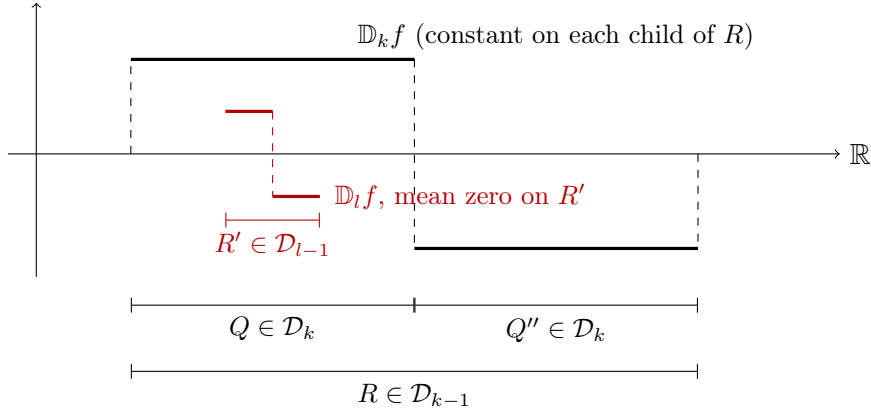
Hence

$$\mathbb{D}_k f(x) = \sum_{R \in \mathcal{D}_{k-1}} \sum_{\substack{Q \in \mathcal{D}_k \\ Q \subsetneq R}} \langle f \rangle_Q \left(\chi_Q(x) - \frac{1}{2^d} \chi_R(x) \right). \quad (1)$$

Two things to read off from (1): the term belonging to R is supported in R , and

$$\int_R \left(\chi_Q(x) - \frac{1}{2^d} \chi_R(x) \right) dx = |Q| - \frac{1}{2^d} |R| = 0, \quad (2)$$

so $\mathbb{D}_k f$ has mean zero on every $R \in \mathcal{D}_{k-1}$.



$d = 1, l > k$. On $R \in \mathcal{D}_{k-1}$ the function $\mathbb{D}_k f$ takes one value on each child Q , with mean zero over R by (2). A cube $R' \in \mathcal{D}_{l-1}$ meeting R is strictly smaller, so it sits inside one child Q , where $\mathbb{D}_k f$ is constant, while $\mathbb{D}_l f$ has mean zero on R' . So the product integrates to zero over R' . In $d = 1$, $\chi_Q - \frac{1}{2} \chi_R$ is a multiple of the Haar function of R .

Now let $k, l \in \mathbb{Z}$ with $l > k$. Using (1) for both $\mathbb{D}_k f$ and $\mathbb{D}_l f$,

$$\begin{aligned} &\int_{\mathbb{R}^d} \mathbb{D}_k f(x) \overline{\mathbb{D}_l f(x)} dx \\ &= \sum_{R \in \mathcal{D}_{k-1}} \sum_{\substack{Q \in \mathcal{D}_k \\ Q \subsetneq R}} \sum_{R' \in \mathcal{D}_{l-1}} \sum_{\substack{Q' \in \mathcal{D}_l \\ Q' \subsetneq R'}} \langle f \rangle_Q \overline{\langle f \rangle_{Q'}} \int_{\mathbb{R}^d} \underbrace{\left(\chi_Q(x) - \frac{1}{2^d} \chi_R(x) \right)}_{\text{supp} \subseteq R} \underbrace{\left(\chi_{Q'}(x) - \frac{1}{2^d} \chi_{R'}(x) \right)}_{\text{supp} \subseteq R'} dx. \end{aligned}$$

The interchange of sum and integral is allowed: for fixed R the inner sum over Q is finite, the terms for different R have disjoint supports, and $\mathbb{D}_k f \overline{\mathbb{D}_l f} \in L^1$, so this is countable additivity of the integral over the partition $\mathcal{D}_{k-1}$, and then over $\mathcal{D}_{l-1}$ inside each R .

Since $l > k$ we have $\ell(R) = 2^{-(k-1)} > 2^{-(l-1)} = \ell(R')$, so for dyadic cubes either $R \cap R' = \emptyset$ or $R' \subsetneq R$. In the first case the integrand vanishes identically, so only the second case matters. Suppose $R' \subsetneq R$. Since R' is dyadic with $\ell(R') \leq 2^{-k}$, there is a child $Q_0 \in \mathcal{D}_k$, $Q_0 \subsetneq R$, with $R' \subseteq Q_0$, and therefore for every child Q of R the function $\chi_Q - 2^{-d}\chi_R$ is constant on R' , equal to $1 - 2^{-d}$ if $Q = Q_0$ and to -2^{-d} otherwise. Call this constant $c_{R'}$. Then

$$\begin{aligned} & \int_{\mathbb{R}^d} \left(\chi_Q(x) - \frac{1}{2^d} \chi_R(x) \right) \left(\chi_{Q'}(x) - \frac{1}{2^d} \chi_{R'}(x) \right) dx \\ &= \int_{R'} \underbrace{\left(\chi_Q(x) - \frac{1}{2^d} \chi_R(x) \right)}_{= c_{R'} \text{ on } R'} \left(\chi_{Q'}(x) - \frac{1}{2^d} \chi_{R'}(x) \right) dx \\ &= c_{R'} \int_{R'} \left(\chi_{Q'}(x) - \frac{1}{2^d} \chi_{R'}(x) \right) dx \quad \stackrel{\substack{= \\ \uparrow \\ (2) \text{ for } R'}}{=} c_{R'} \left(|Q'| - \frac{1}{2^d} |R'| \right) = 0. \end{aligned}$$

Hence every term vanishes and

$$\langle \mathbb{D}_k f, \mathbb{D}_l f \rangle = \int_{\mathbb{R}^d} \mathbb{D}_k f(x) \overline{\mathbb{D}_l f(x)} dx = 0 \quad \text{for } k \neq l.$$

The $\mathbb{D}_k f$ are mutually orthogonal in $L^2(\mathbb{R}^d)$, so by Pythagoras and (a),

$$\begin{aligned} \|f\|_{L^2}^2 & \stackrel{\substack{= \\ \uparrow \\ (a), L^2 \text{ convergence}}}{=} \lim_{N, M \rightarrow \infty} \left\| \sum_{k=-N}^M \mathbb{D}_k f \right\|_{L^2}^2 \\ & \stackrel{\substack{= \\ \uparrow \\ \text{orthogonality}}}{=} \lim_{N, M \rightarrow \infty} \sum_{k=-N}^M \|\mathbb{D}_k f\|_{L^2}^2 = \sum_{k=-\infty}^{\infty} \|\mathbb{D}_k f\|_{L^2}^2. \quad \blacksquare \end{aligned}$$

What this says. $L^2(\mathbb{R}^d)$ splits as an orthogonal direct sum of the spaces $\{\mathbb{D}_k f\}$, one for each scale 2^{-k} , and every f is the sum of its pieces at the different scales. This is the dyadic counterpart of Plancherel, with the scale k playing the role of the frequency band $|\xi| \sim 2^k$, and it is the dyadic version of the identity $\int_{\mathbb{R}_+^{d+1}} |f * \psi_t|^2 dx \frac{dt}{t} \simeq \|f\|_{L^2}^2$ that sits behind the Carleson measure theorem for BMO in Lecture 13. The square function $(\sum_k |\mathbb{D}_k f|^2)^{1/2}$ is next.