

Harmonic Analysis

Lecture 15

Teodoras Paura

23 October 2026

Exercise Classes: Thursdays 13.00–15.00

Dyadic maximal function and dyadic H^1

Close link to probability theory, like the conditional expectations of the last lecture.

Definition. The *dyadic (martingale) maximal function* of $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ is defined as

$$M_{\mathcal{D}}f := \sup_{k \in \mathbb{Z}} |\mathbb{E}_k f|.$$

Recall $\mathbb{E}_k f = \sum_{Q \in \mathcal{D}, \ell(Q)=2^{-k}} \chi_Q \langle f \rangle_Q$, so

$$M_{\mathcal{D}}f(x) = \sup_{\substack{Q \in \mathcal{D} \\ x \in Q}} \frac{1}{|Q|} \left| \int_Q f(y) dy \right|,$$

the regular (uncentred) maximal function, but only over dyadic cubes. Also $(\mathbb{E}_k f)_{k \in \mathbb{Z}}$ is a dyadic martingale, and $\mathbb{E}_k f \rightarrow f$ as $k \rightarrow \infty$ in L^2 and a.e. (Lecture 14).

Two differences from the Hardy–Littlewood maximal function: the absolute value is outside the average, so cancellation inside a cube helps, and only the countably many dyadic cubes containing x compete, one for each scale.

Theorem 1. For every $p \in (1, \infty]$ there exists $C_{p,d} > 0$ such that

$$\|f\|_{L^p(\mathbb{R}^d)} \leq \|M_{\mathcal{D}}f\|_{L^p(\mathbb{R}^d)} \leq C_{p,d} \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in L^p(\mathbb{R}^d),$$

and one can take $C_{p,d} = \frac{p}{p-1}$ for $p < \infty$ and $C_{\infty,d} = 1$.

Proof. *Lower bound.* By Lebesgue differentiation along dyadic cubes (Assignment 1, Question 2(c)), $|f(x)| = \lim_{k \rightarrow \infty} |\mathbb{E}_k f(x)| \leq M_{\mathcal{D}}f(x)$ for a.e. x . *Case $p = \infty$.* $|\langle f \rangle_Q| \leq \|f\|_{L^\infty}$ for every Q , so $\|M_{\mathcal{D}}f\|_{L^\infty} \leq \|f\|_{L^\infty}$.

Weak type (1,1), via the Calderón–Zygmund stopping cubes. Let $f \in L^1(\mathbb{R}^d)$ and $a > 0$. If $M_{\mathcal{D}}f(x) > a$ there is a dyadic $Q \ni x$ with $|\langle f \rangle_Q| > a$, and since $|\langle f \rangle_Q| \leq \|f\|_{L^1}/|Q| \rightarrow 0$ as Q grows, there is a largest such cube. Let $\mathcal{Q}$ be the family of these maximal cubes,

$$\mathcal{Q} = \{Q \in \mathcal{D} : |\langle f \rangle_Q| > a, |\langle f \rangle_{Q'}| \leq a \text{ for the dyadic parent } Q' \text{ of } Q\}.$$

Two maximal dyadic cubes are either equal or disjoint, so $\mathcal{Q}$ is a disjoint family, and $\{x : M_{\mathcal{D}}f(x) > a\} = \bigcup_{Q \in \mathcal{Q}} Q$. Hence

$$\begin{aligned} |\{x \in \mathbb{R}^d : M_{\mathcal{D}}f(x) > a\}| &= \sum_{Q \in \mathcal{Q}} |Q| \leq \sum_{\substack{a|Q| < \int_Q |f| \\ \uparrow \\ a|Q| < \int_Q |f|}} |Q| \leq \frac{1}{a} \sum_{Q \in \mathcal{Q}} \int_Q |f(x)| dx \\ &= \frac{1}{a} \int_{\{M_{\mathcal{D}}f > a\}} |f(x)| dx \leq \frac{\|f\|_{L^1}}{a}. \end{aligned} \tag{*}$$

This is weak type $(1, 1)$ with constant 1, and the same argument works verbatim for $f \in L^p$, $p < \infty$, since then $|\langle f \rangle_Q| \leq |Q|^{-1/p} \|f\|_{L^p} \rightarrow 0$ as well.

Strong type (p, p) . Now interpolate with the Marcinkiewicz interpolation theorem between $p_0 = 1$ and $p_1 = \infty$, with constants $A = B = 1$ in the notation of the previous lectures. Keeping the middle expression in $(\dagger)$ instead of the last one gives the sharp constant: by the layer cake formula and Tonelli, for $f \in L^1 \cap L^\infty$ (so that $\|M_{\mathcal{D}}f\|_{L^p} < \infty$),

$$\begin{aligned} \|M_{\mathcal{D}}f\|_{L^p}^p &= \int_0^\infty p a^{p-1} |\{M_{\mathcal{D}}f > a\}| da \stackrel{(\dagger)}{\leq} \int_0^\infty p a^{p-2} \int_{\{M_{\mathcal{D}}f > a\}} |f(x)| dx da \\ &\stackrel{\text{Tonelli}}{=} p \int_{\mathbb{R}^d} |f(x)| \int_0^{M_{\mathcal{D}}f(x)} a^{p-2} da dx \\ &= \frac{p}{p-1} \int_{\mathbb{R}^d} |f(x)| (M_{\mathcal{D}}f(x))^{p-1} dx \stackrel{\text{H\"older, } q = \frac{p}{p-1}}{\leq} \frac{p}{p-1} \|f\|_{L^p} \|M_{\mathcal{D}}f\|_{L^p}^{p-1}, \end{aligned}$$

and dividing by $\|M_{\mathcal{D}}f\|_{L^p}^{p-1}$ gives $\|M_{\mathcal{D}}f\|_{L^p} \leq \frac{p}{p-1} \|f\|_{L^p}$. For general $f \in L^p$ apply this to $|f| \chi_{\{|f| \leq n\} \cap B(0, n)}$ and let $n \rightarrow \infty$ by monotone convergence, using $M_{\mathcal{D}}f \leq M_{\mathcal{D}}|f|$. ■

The constant $\frac{p}{p-1}$ is Doob's maximal inequality for martingales and is sharp. The generic Marcinkiewicz bound from Lecture 2 with $A = B = 1$ would give $2(\frac{p}{p-1})^{1/p}$; the improvement comes from the middle term in $(\dagger)$, which says the level set of $M_{\mathcal{D}}f$ is controlled by the mass of f on that level set, not by all of $\|f\|_{L^1}$.

Definition. The dyadic Hardy space $H_{\mathcal{D}}^1(\mathbb{R}^d)$ is given as

$$H_{\mathcal{D}}^1(\mathbb{R}^d) := \{f \in L^1(\mathbb{R}^d) : M_{\mathcal{D}}f \in L^1(\mathbb{R}^d)\}, \quad \|f\|_{H_{\mathcal{D}}^1} := \|M_{\mathcal{D}}f\|_{L^1(\mathbb{R}^d)}.$$

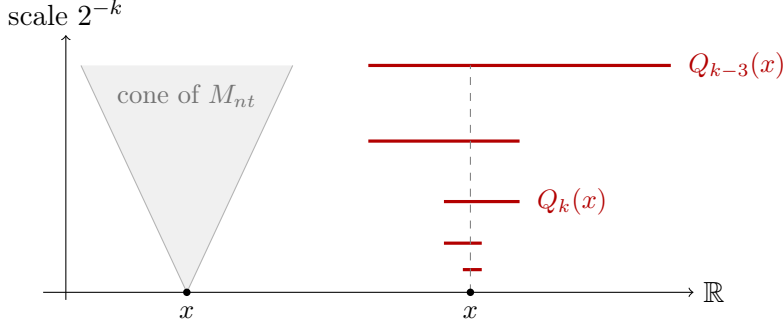
Recall the theorem about $H^1(\mathbb{R}^d)$ (Lecture 9): for $f \in L^1(\mathbb{R}^d)$ the following are equivalent.

- (a) $f \in H^1(\mathbb{R}^d)$.
- (b) $M_{nt}f \in L^1(\mathbb{R}^d)$, where $M_{nt}f(x) = \sup_{|y-x| < t} |(P_t * f)(y)|$ is the nontangential maximal function.
- (c) $Sf \in L^1(\mathbb{R}^d)$, where S is the square function.
- (d) $f = \sum_k \alpha_k a_{I_k}$ with $(\alpha_k) \in \ell^1(\mathbb{N})$ and a_{I_k} atoms.

Theorem 2. Let $f \in L^1(\mathbb{R}^d)$. Then the following are equivalent.

- (a)+(b) $f \in H_{\mathcal{D}}^1(\mathbb{R}^d)$, that is $M_{\mathcal{D}}f \in L^1(\mathbb{R}^d)$.
- (c) $S_{\mathcal{D}}f \in L^1(\mathbb{R}^d)$, with the dyadic square function $S_{\mathcal{D}}$ defined below.
- (d) $f = \sum_k \alpha_k a_{I_k}$ with $(\alpha_k) \in \ell^1(\mathbb{N})$ and a_{I_k} dyadic atoms: a_{I_k} supported in a dyadic cube I_k , $\int a_{I_k} = 0$, $\|a_{I_k}\|_{L^\infty} \leq |I_k|^{-1}$.

(a) and (b) merge into one condition: recall the picture of M_{nt} , a cone above x over which the Poisson averages are taken. For $M_{\mathcal{D}}$ the picture is the tower of dyadic cubes containing x , one at each scale. Note it is not centred at x , but it is some sort of cone too, and this is the connection between the dyadic and the nontangential maximal function.



$d = 1$. Left: $M_{nt}f(x)$ takes the supremum of $|P_t * f(y)|$ over the cone $|y - x| < t$. Right: $M_{\mathcal{D}}f(x)$ takes the supremum of $|\langle f \rangle_Q|$ over the dyadic intervals $Q_k(x) \ni x$, drawn at height equal to their length. The tower is a jagged, off-centre cone with vertex x .

Definition. For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ we define the *dyadic (martingale) square function*

$$S_{\mathcal{D}}f := \left(\sum_{k \in \mathbb{Z}} |\mathbb{D}_k f|^2 \right)^{1/2}, \quad \mathbb{D}_k f = \mathbb{E}_k f - \mathbb{E}_{k-1} f.$$

The $\mathbb{D}_k f$ are the Haar frequency packages: $\mathbb{D}_k f$ collects the Haar coefficients of f at scale 2^{-k} , the dyadic analogue of the frequency band $|\xi| \sim 2^k$.

Recall the Littlewood–Paley square function

$$Sf = \left(\sum_k |\check{\delta}_k * f|^2 \right)^{1/2},$$

where $\check{\delta}_k * f$ is the piece of f with frequencies $|\xi| \sim 2^k$, that is $\widehat{\check{\delta}_k * f} = \delta_k \widehat{f}$ with δ_k the cut-off to the k -th dyadic frequency band. Theorem 2 is not proved today; we first prove the L^p theory of $S_{\mathcal{D}}$, which is the dyadic version of the Littlewood–Paley theorem.

Theorem 3. For every $p \in (1, \infty)$ there exist constants $A_{p,d}, B_{p,d} > 0$ such that

$$A_{p,d} \|f\|_{L^p(\mathbb{R}^d)} \leq \|S_{\mathcal{D}}f\|_{L^p(\mathbb{R}^d)} \leq B_{p,d} \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in L^p(\mathbb{R}^d).$$

Proof of the upper bound. Again let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\{r_k\}_{k \in \mathbb{Z}}$ Rademacher functions on $(\Omega, \mathcal{A}, \mathbb{P})$, that is independent random variables with $\mathbb{P}(r_k = 1) = \mathbb{P}(r_k = -1) = \frac{1}{2}$. Let $N \in \mathbb{N}$ and $\omega \in \Omega$, and define the *Haar multiplier*

$$T_{\omega, N} f := \sum_{k=-N}^N r_k(\omega) \mathbb{D}_k f.$$

We know $f = \sum_{k=-\infty}^{\infty} \mathbb{D}_k f$ with convergence a.e. and in L^p for $f \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$ (Lecture 14; the proof given there for $p = 2$ works for every $p < \infty$). So $T_{\omega, N}$ multiplies each dyadic piece of f by a constant, exactly as a Fourier multiplier does with the frequency pieces:

$$f \text{ “} = \text{”} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi \longrightarrow \int_{\mathbb{R}^d} m(\xi) \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi,$$

and $r_k(\omega)$ for a fixed ω is a number depending only on k , so $T_{\omega, N}$ is the Haar analogue with $m(k) = r_k(\omega) = \pm 1$.

Claim. For each $p \in (1, \infty)$ there exists $C_{p,d} > 0$, depending only on p and d , such that

$$\|T_{\omega,N}f\|_{L^p(\mathbb{R}^d)} \leq C_{p,d} \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in L^p(\mathbb{R}^d), \omega \in \Omega, N \in \mathbb{N}.$$

The point is that the constant is uniform in ω and N .

Case 1: $1 < p \leq 2$. First $p = 2$. By Pythagoras, since the $\mathbb{D}_k f$ are mutually orthogonal (Lecture 14) and $|r_k(\omega)| = 1$,

$$\begin{aligned} \|T_{\omega,N}f\|_{L^2}^2 &= \left\| \sum_{k=-N}^N r_k(\omega) \mathbb{D}_k f \right\|_{L^2}^2 = \sum_{k=-N}^N |r_k(\omega)|^2 \|\mathbb{D}_k f\|_{L^2}^2 \\ &\leq \sum_{k=-\infty}^{\infty} \|\mathbb{D}_k f\|_{L^2}^2 \stackrel{\substack{\uparrow \\ \text{Lecture 14, (b)}}}{=} \|f\|_{L^2}^2. \end{aligned}$$

So $T_{\omega,N}$ is of strong type $(2, 2)$ with constant 1.

Now $p = 1$, weak type. Let $f \in L^1(\mathbb{R}^d)$ and $a > 0$. We consider $|\{x \in \mathbb{R}^d : |T_{\omega,N}f(x)| > a\}|$. Take the Calderón–Zygmund decomposition of f at height a (Lecture 5): a family $\mathcal{Q}$ of mutually disjoint dyadic cubes with

$$\sum_{Q \in \mathcal{Q}} |Q| \leq \frac{1}{a} \|f\|_{L^1}, \quad b := \sum_{Q \in \mathcal{Q}} (f - \langle f \rangle_Q) \chi_Q, \quad \text{supp } b \subseteq \bigcup_{Q \in \mathcal{Q}} Q,$$

$$g := f - b, \quad |g(x)| \leq 2^d a.$$

$g = f$ off $\bigcup \mathcal{Q}$, where $|f| \leq a$ a.e., and $g = \langle f \rangle_Q$ on $Q \in \mathcal{Q}$, where $a < |\langle f \rangle_Q| \leq 2^d a$ by maximality. In particular $\|g\|_{L^1} \leq \|f\|_{L^1}$. Since $T_{\omega,N}f = T_{\omega,N}g + T_{\omega,N}b$,

$$|\{x \in \mathbb{R}^d : |T_{\omega,N}f(x)| > a\}| \leq \underbrace{|\{x \in \mathbb{R}^d : |T_{\omega,N}g(x)| > \frac{a}{2}\}|}_{=:(I)} + \underbrace{|\{x \in \mathbb{R}^d : |T_{\omega,N}b(x)| > \frac{a}{2}\}|}_{=:(II)}.$$

The good part (I). Recall that $\|g\|_{L^2}^2 \leq \|g\|_{L^\infty} \|g\|_{L^1} \leq 2^d a \|f\|_{L^1}$. Hence

$$(I) \quad \underbrace{\leq}_{\substack{\uparrow \\ \text{Chebyshev}}} \frac{4}{a^2} \|T_{\omega,N}g\|_{L^2}^2 \leq \underbrace{\leq}_{\substack{\uparrow \\ \text{strong type (2,2)}}} \frac{4}{a^2} \|g\|_{L^2}^2 \leq \frac{4}{a^2} 2^d a \|f\|_{L^1} = \frac{2^{d+2}}{a} \|f\|_{L^1}.$$

So (I) is fine. Now (II).

The bad part (II). Claim: $\text{supp } T_{\omega,N}b \subseteq \bigcup_{Q \in \mathcal{Q}} Q$. Recall

$$b = \sum_{Q \in \mathcal{Q}} b_Q, \quad b_Q := \chi_Q (f - \langle f \rangle_Q), \quad \text{supp } b_Q \subseteq Q, \quad \int_Q b_Q = 0.$$

We will show $\text{supp } T_{\omega,N}b_Q \subseteq Q$ for each $Q \in \mathcal{Q}$. Let $\ell(Q) = 2^{-k}$.

Step 1: $\mathbb{E}_l b_Q = 0$ for $l \leq k$. For $l = k$: $\langle b_Q \rangle_Q = 0$, and for $Q' \in \mathcal{D}$ with $\ell(Q') = \ell(Q) = 2^{-k}$, $Q' \neq Q$, we have $Q' \cap Q = \emptyset$, so $\langle b_Q \rangle_{Q'} = 0$. Now let $Q' \in \mathcal{D}$ with $\ell(Q') = 2^{-l}$, $l < k$, so Q' is larger than Q . Then either $Q \subseteq Q'$, in which case

$$\langle b_Q \rangle_{Q'} = \frac{1}{|Q'|} \int_{Q'} b_Q = \frac{|Q|}{|Q'|} \langle b_Q \rangle_Q = 0,$$

or $Q \cap Q' = \emptyset$, in which case $\langle b_Q \rangle_{Q'} = 0$ trivially. Hence $\mathbb{E}_l b_Q = \sum_{Q' \in \mathcal{D}_l} \chi_{Q'} \langle b_Q \rangle_{Q'} = 0$ for all $l \leq k$.

Step 2: $\text{supp } \mathbb{E}_l b_Q \subseteq Q$ for $l > k$. Now the cubes $Q' \in \mathcal{D}_l$ are smaller than Q , so again only two possibilities: $Q' \cap Q = \emptyset$, and then $\langle b_Q \rangle_{Q'} = 0$, or $Q' \subseteq Q$. Thus

$$\mathbb{E}_l b_Q = \sum_{\substack{Q' \in \mathcal{D}_l \\ Q' \subseteq Q}} \chi_{Q'} \langle b_Q \rangle_{Q'}, \quad \text{so} \quad \text{supp } \mathbb{E}_l b_Q \subseteq Q \quad \text{for } l > k.$$

Consequently $\text{supp } \mathbb{D}_l b_Q = \text{supp}(\mathbb{E}_l b_Q - \mathbb{E}_{l-1} b_Q) \subseteq Q$ for $l > k$ (for $l = k + 1$ the second term is 0 by Step 1), and by telescoping

$$b_Q = \sum_{l=k+1}^{\infty} \mathbb{D}_l b_Q,$$

each term supported in Q . *Telescoping:* $\sum_{l=k+1}^L \mathbb{D}_l b_Q = \mathbb{E}_L b_Q - \mathbb{E}_k b_Q = \mathbb{E}_L b_Q \rightarrow b_Q$ a.e. as $L \rightarrow \infty$.

Step 3. Since $\mathbb{D}_l b_Q = 0$ for $l \leq k$,

$$T_{\omega, N} b_Q = \sum_{l=\max(-N, k+1)}^N r_l(\omega) \mathbb{D}_l b_Q,$$

a finite sum of functions supported in Q , so $\text{supp } T_{\omega, N} b_Q \subseteq Q$. Thus the sum $T_{\omega, N} b = \sum_{Q \in \mathcal{Q}} T_{\omega, N} b_Q$ is also supported in $\bigcup_{Q \in \mathcal{Q}} Q$, and the Claim holds. $T_{\omega, N} b = \sum_Q T_{\omega, N} b_Q$ because each $\mathbb{E}_l$ commutes with the sum over $\mathcal{Q}$: the b_Q have disjoint supports and $b \in L^1$, so $\langle b \rangle_{Q'} = \sum_Q \langle b_Q \rangle_{Q'}$ for every cube Q' .

Concluding the weak type estimate. By the Claim, $\{|T_{\omega, N} b| > \frac{a}{2}\} \subseteq \bigcup_{Q \in \mathcal{Q}} Q$, so

$$(II) \leq \left| \bigcup_{Q \in \mathcal{Q}} Q \right| = \sum_{Q \in \mathcal{Q}} |Q| \leq \frac{1}{a} \|f\|_{L^1},$$

and altogether

$$|\{x \in \mathbb{R}^d : |T_{\omega, N} f(x)| > a\}| \leq (I) + (II) \leq \frac{2^{d+2} + 1}{a} \|f\|_{L^1(\mathbb{R}^d)}$$

for all $f \in L^1$, $a > 0$, $\omega \in \Omega$, $N \in \mathbb{N}$. So $T_{\omega, N}$ is of weak type $(1, 1)$ and strong type $(2, 2)$, with constants independent of ω and N , and the Marcinkiewicz interpolation theorem gives the Claim for $1 < p \leq 2$.

Where this is going (to be completed next lecture). Case 2, $2 < p < \infty$: $T_{\omega, N}$ is self-adjoint on L^2 , because each $\mathbb{D}_k$ is an orthogonal projection and $r_k(\omega)$ is real, so the bound for p follows from the bound for the conjugate exponent $q \in (1, 2)$ by duality. Then Khintchine's inequality, $\mathbb{E}_\omega |\sum_k r_k(\omega) c_k|^p \simeq_p (\sum_k |c_k|^2)^{p/2}$, applied at each fixed x with $c_k = \mathbb{D}_k f(x)$, turns the uniform bound on $T_{\omega, N}$ into the square function bound: $\int_{\mathbb{R}^d} (\sum_{|k| \leq N} |\mathbb{D}_k f|^2)^{p/2} dx \lesssim_p \mathbb{E}_\omega \|T_{\omega, N} f\|_{L^p}^p \leq C_{p,d}^p \|f\|_{L^p}^p$, and $N \rightarrow \infty$ by monotone convergence gives the upper bound in Theorem 3. The lower bound is duality again: $\int f \bar{g} = \sum_k \int \mathbb{D}_k f \overline{\mathbb{D}_k g} \leq \int S_{\mathcal{D}} f S_{\mathcal{D}} g \leq \|S_{\mathcal{D}} f\|_{L^p} B_{q,d} \|g\|_{L^q}$, and taking the supremum over $\|g\|_{L^q} \leq 1$ gives $\|f\|_{L^p} \leq B_{q,d} \|S_{\mathcal{D}} f\|_{L^p}$. The whole proof is the Calderón–Zygmund argument from the Hilbert transform, with the single new input that dyadic martingale differences of a function with mean zero on a dyadic cube Q never leave Q : there is no tail outside $2Q$ to estimate, which is why the dyadic theory is cleaner than the continuous one.