

Harmonic Analysis

Lecture 16

Teodoras Paura

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Exercise Classes: Thursdays 13.00–15.00

Back to where we stopped in Lecture 15. We had the dyadic square function

$$S_{\mathcal{D}}f(x) := \left(\sum_{k \in \mathbb{Z}} |\mathbb{D}_k f(x)|^2 \right)^{1/2}, \quad \mathbb{D}_k f = \mathbb{E}_k f - \mathbb{E}_{k-1} f,$$

and we know already that

$$\|S_{\mathcal{D}}f\|_{L^2(\mathbb{R}^d)}^2 = \sum_{k \in \mathbb{Z}} \|\mathbb{D}_k f\|_{L^2}^2 = \|f\|_{L^2}^2 \quad (\text{Lecture 14}).$$

Theorem. For $1 < p < \infty$ there exist $A_{p,d}, B_{p,d} > 0$ such that

$$A_{p,d} \|f\|_{L^p(\mathbb{R}^d)} \leq \|S_{\mathcal{D}}f\|_{L^p(\mathbb{R}^d)} \leq B_{p,d} \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in L^p(\mathbb{R}^d). \quad (*)$$

What we have for $1 < p \leq 2$. With Rademacher functions $\{r_k\}_{k \in \mathbb{Z}}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and the Haar multiplier

$$T_{\omega,N}f = \sum_{k=-N}^N r_k(\omega) \mathbb{D}_k f,$$

we showed $\|T_{\omega,N}f\|_{L^2(\mathbb{R}^d)} \leq \|f\|_{L^2(\mathbb{R}^d)}$ for all $N \in \mathbb{N}$, $\omega \in \Omega$, and with the Calderón-Zygmund decomposition $f = g + b$ of f at height a , $b = \sum_{Q \in \mathcal{Q}} b_Q = \sum_{Q \in \mathcal{Q}} \chi_Q(f - \langle f \rangle_Q)$,

$$\begin{aligned} |\{|T_{\omega,N}f| > a\}| &\leq \underbrace{|\{|T_{\omega,N}g| > \frac{a}{2}\}|}_{\leq C \frac{\|f\|_{L^1}}{a}} + \underbrace{|\{|T_{\omega,N}b| > \frac{a}{2}\}|}_{\leq \sum_{Q \in \mathcal{Q}} |Q| \leq \frac{\|f\|_{L^1}}{a}} \leq C \frac{\|f\|_{L^1}}{a} + \frac{\|f\|_{L^1}}{a}, \end{aligned}$$

where the second bound came from $\text{supp } T_{\omega,N}b_Q \subseteq Q$. Now we employ the Marcinkiewicz interpolation theorem and obtain

$$\|T_{\omega,N}\|_{L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)} \leq \frac{C_d}{p-1} \quad \text{for } p \in (1, 2], \text{ uniformly in } \omega \in \Omega, N \in \mathbb{N}. \quad (1)$$

Marcinkiewicz between weak $(1,1)$ and strong $(2,2)$ gives a constant that blows up at both ends. The blow-up at $p = 2$ is spurious here: $T_{\omega,N}$ is linear with $\|T_{\omega,N}\|_{L^2 \rightarrow L^2} \leq 1$, so Riesz-Thorin between $p = \frac{3}{2}$ and $p = 2$ gives a bound uniform on $[\frac{3}{2}, 2]$, and only the $\frac{1}{p-1}$ from the $p \rightarrow 1$ end survives.

Consider $2 \leq p < \infty$. Note that

$$\mathbb{E}_k : L^2(\mathbb{R}^d) \longrightarrow V_k := \overline{\text{span}\{\chi_Q : Q \in \mathcal{D}_k\}}^{\|\cdot\|_{L^2}}$$

is the orthogonal projection onto V_k . Recall: for $A : \mathcal{H} \rightarrow \mathcal{H}$ a bounded linear operator on a Hilbert space, A is self-adjoint if and only if $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in \mathcal{H}$. Let $f \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^d) \cap L^{p'}(\mathbb{R}^d)$. Then

$$\begin{aligned} \langle \mathbb{E}_k f, g \rangle_{L^2} &= \left\langle \sum_{Q \in \mathcal{D}_k} \chi_Q \langle f \rangle_Q, g \right\rangle_{L^2} = \sum_{Q \in \mathcal{D}_k} \langle \langle f \rangle_Q, g \rangle_{L^2(Q)} = \sum_{Q \in \mathcal{D}_k} |Q| \langle f \rangle_Q \overline{\langle g \rangle_Q} \\ &= \sum_{Q \in \mathcal{D}_k} \langle f, \langle g \rangle_Q \rangle_{L^2(Q)} = \langle f, \mathbb{E}_k g \rangle_{L^2}. \end{aligned}$$

Hence

$$\langle \mathbb{D}_k f, g \rangle = \langle \mathbb{E}_k f - \mathbb{E}_{k-1} f, g \rangle = \langle f, \mathbb{E}_k g \rangle - \langle f, \mathbb{E}_{k-1} g \rangle = \langle f, \mathbb{D}_k g \rangle,$$

so each $\mathbb{D}_k$ is self-adjoint, and since $r_k(\omega) \in \mathbb{R}$, so is $T_{\omega, N} = \sum_{k=-N}^N r_k(\omega) \mathbb{D}_k$.

Now let $2 \leq p < \infty$, $f \in L^2 \cap L^p$, $g \in L^2 \cap L^{p'}$ with $\|g\|_{L^{p'}} = 1$. Then, using that $p' \in (1, 2]$ and (1) for p' ,

$$\left| \langle T_{\omega, N} f, g \rangle_{L^2} \right| \underset{\substack{\uparrow \\ \text{self-adjoint}}}{=} \left| \langle f, T_{\omega, N} g \rangle_{L^2} \right| \underset{\substack{\uparrow \\ \text{H\"older}}}{\leq} \|f\|_{L^p} \|T_{\omega, N} g\|_{L^{p'}} \underset{\substack{\uparrow \\ (1), \|g\|_{L^{p'}} = 1}}{\leq} \|f\|_{L^p} \frac{C_d}{p' - 1}.$$

Since $L^2(\mathbb{R}^d) \cap L^{p'}(\mathbb{R}^d)$ is dense in $L^{p'}(\mathbb{R}^d)$ and both sides are continuous in g for the $L^{p'}$ norm, this gives

$$\left| \langle T_{\omega, N} f, g \rangle \right| \leq \frac{C_d}{p' - 1} \|f\|_{L^p} \quad \text{for all } g \in L^{p'}(\mathbb{R}^d) \text{ with } \|g\|_{L^{p'}} = 1,$$

and therefore, by duality,

$$\|T_{\omega, N} f\|_{L^p(\mathbb{R}^d)} \leq \frac{C_d}{p' - 1} \|f\|_{L^p(\mathbb{R}^d)} = C_d (p - 1) \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for } 2 \leq p < \infty, \quad (2)$$

first for $f \in L^2 \cap L^p$ and then for all $f \in L^p$ by density. Here we have used a duality: for X a Banach space with dual X^* ,

$$\|f\|_X = \sup_{\substack{g \in X^* \\ \|g\|_{X^*} = 1}} |\langle f, g \rangle_{X, X^*}|,$$

the supremum of the linear functionals g acting on f . For $X = L^p$, $X^* = L^{p'}$ and the pairing is $\int f \bar{g}$. The lower bound from the easy direction is H\"older, the upper bound is Hahn-Banach, or for L^p simply the choice $g = |f|^{p-1} \overline{\text{sgn } f} / \|f\|_{L^p}^{p-1}$.

Together, (1) and (2) say: for each $p \in (1, \infty)$,

$$\|T_{\omega, N} f\|_{L^p(\mathbb{R}^d)} \leq C_{p,d} \|f\|_{L^p(\mathbb{R}^d)}, \quad C_{p,d} := C_d \max\left\{ \frac{1}{p-1}, \frac{1}{p'-1} \right\},$$

for all $f \in L^p$, $\omega \in \Omega$, $N \in \mathbb{N}$.

Upper bound in (*). We use Khintchine's inequality: for each $p \in [1, \infty)$ there is a constant $A'_p > 0$ such that for every finite sequence (c_k) of complex numbers

$$\left(\sum_k |c_k|^2 \right)^{p/2} \leq A'^p_p \int_{\Omega} \left| \sum_k c_k r_k(\omega) \right|^p d\mathbb{P}(\omega).$$

Recall $(\Omega, \mathcal{A}, \mathbb{P})$ is where the Rademacher functions r_k live: independent, $\mathbb{P}(r_k = \pm 1) = \frac{1}{2}$. Khintchine says that for random signs, all $L^p(\Omega)$ norms of $\sum c_k r_k$ are comparable to the ℓ^2 norm of the coefficients. The reverse inequality holds too, with another constant, but is not needed here.

Hence for each $N \in \mathbb{N}$ and $p \in (1, \infty)$, writing $S_{\mathcal{D},N}f(x) := (\sum_{k=-N}^N |\mathbb{D}_k f(x)|^2)^{1/2}$ and applying Khintchine at each fixed x with $c_k = \mathbb{D}_k f(x)$,

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\sum_{k=-N}^N |\mathbb{D}_k f(x)|^2 \right)^{p/2} dx &\stackrel{\substack{\uparrow \\ \text{Khintchine}}}{\leq} A_p'^p \int_{\mathbb{R}^d} \int_{\Omega} \left| \sum_{k=-N}^N r_k(\omega) \mathbb{D}_k f(x) \right|^p d\mathbb{P}(\omega) dx \\ &\stackrel{\substack{\uparrow \\ \text{Fubini}}}{=} A_p'^p \int_{\Omega} \underbrace{\int_{\mathbb{R}^d} |T_{\omega,N}f(x)|^p dx}_{\leq C_{p,d}^p \|f\|_{L^p}^p \text{ for every } \omega} d\mathbb{P}(\omega) \leq A_p'^p C_{p,d}^p \|f\|_{L^p}^p. \end{aligned}$$

Fubini applies since $(x, \omega) \mapsto T_{\omega,N}f(x)$ is a finite sum of products of measurable functions, and $\mathbb{P}(\Omega) = 1$ is what makes the uniform bound in ω survive the integration. Since $S_{\mathcal{D},N}f \uparrow S_{\mathcal{D}}f$ pointwise as $N \rightarrow \infty$, by the monotone convergence theorem

$$\|S_{\mathcal{D}}f\|_{L^p(\mathbb{R}^d)} = \lim_{N \rightarrow \infty} \|S_{\mathcal{D},N}f\|_{L^p(\mathbb{R}^d)} \leq \underbrace{A_p' C_d \max\left\{\frac{1}{p-1}, \frac{1}{p'-1}\right\}}_{=: B_{p,d}} \|f\|_{L^p(\mathbb{R}^d)}.$$

This is the upper bound in (*), for all $f \in L^p(\mathbb{R}^d)$.

Lower bound in (*). Let first $f \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. As used before,

$$\|f\|_{L^p(\mathbb{R}^d)} = \sup_{\substack{g \in L^{p'}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \\ \|g\|_{L^{p'}}=1}} |\langle f, g \rangle_{L^2}|,$$

where the supremum may be restricted to $L^2 \cap L^{p'}$ by density. For such g , expanding both f and g in martingale differences (Lecture 14, with convergence in L^2),

$$\begin{aligned} |\langle f, g \rangle_{L^2}| &= \left| \left\langle \sum_{k=-\infty}^{\infty} \mathbb{D}_k f, \sum_{l=-\infty}^{\infty} \mathbb{D}_l g \right\rangle_{L^2} \right| \stackrel{\substack{\uparrow \\ L^2 \text{ convergence}}}{=} \left| \sum_k \sum_l \langle \mathbb{D}_k f, \mathbb{D}_l g \rangle_{L^2} \right| \\ &\stackrel{\substack{\uparrow \\ \text{orthogonality}}}{=} \left| \sum_{k=-\infty}^{\infty} \langle \mathbb{D}_k f, \mathbb{D}_k g \rangle_{L^2(\mathbb{R}^d)} \right| = \left| \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^d} \mathbb{D}_k f(x) \overline{\mathbb{D}_k g(x)} dx \right| \\ &\stackrel{\substack{\uparrow \\ \text{move the series inside}}}{=} \left| \int_{\mathbb{R}^d} \sum_{k=-\infty}^{\infty} \mathbb{D}_k f(x) \overline{\mathbb{D}_k g(x)} dx \right| \leq \int_{\mathbb{R}^d} \sum_{k=-\infty}^{\infty} |\mathbb{D}_k f(x)| |\mathbb{D}_k g(x)| dx \\ &\stackrel{\substack{\uparrow \\ \text{Cauchy-Schwarz in } \ell^2}}{\leq} \int_{\mathbb{R}^d} \underbrace{\left(\sum_k |\mathbb{D}_k f(x)|^2 \right)^{1/2}}_{S_{\mathcal{D}}f(x)} \underbrace{\left(\sum_k |\mathbb{D}_k g(x)|^2 \right)^{1/2}}_{S_{\mathcal{D}}g(x)} dx \\ &\stackrel{\substack{\uparrow \\ \text{H\"older}}}{\leq} \|S_{\mathcal{D}}f\|_{L^p(\mathbb{R}^d)} \|S_{\mathcal{D}}g\|_{L^{p'}(\mathbb{R}^d)} \stackrel{\substack{\uparrow \\ \text{upper bound for } p'}}{\leq} \|S_{\mathcal{D}}f\|_{L^p(\mathbb{R}^d)} B_{p',d} \|g\|_{L^{p'}(\mathbb{R}^d)} \\ &= B_{p',d} \|S_{\mathcal{D}}f\|_{L^p(\mathbb{R}^d)}. \end{aligned}$$

Orthogonality $\langle \mathbb{D}_k f, \mathbb{D}_l g \rangle = 0$ for $k \neq l$ is the Lecture 14 computation with g in place of the second f , nothing there used that the two functions were equal. Moving the series inside the integral is dominated convergence: the partial sums are dominated by $\sum_k |\mathbb{D}_k f| |\mathbb{D}_k g| \leq S_{\mathcal{D}}f S_{\mathcal{D}}g$, which is in L^1 by the next two lines. Taking the supremum over g ,

$$\frac{1}{B_{p',d}} \|f\|_{L^p(\mathbb{R}^d)} \leq \|S_{\mathcal{D}}f\|_{L^p(\mathbb{R}^d)} \quad \text{for } f \in L^p \cap L^2, \quad (\#)$$

so this concludes $(*)$ on $L^p \cap L^2$ with $A_{p,d} = \frac{1}{B_{p',d}}$.

For general $f \in L^p(\mathbb{R}^d)$, let $\varepsilon > 0$ and choose $h \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ with $\|h - f\|_{L^p} < \varepsilon$. Since $S_{\mathcal{D}}$ is sublinear, $S_{\mathcal{D}}h \leq S_{\mathcal{D}}f + S_{\mathcal{D}}(h - f)$ pointwise (Minkowski in ℓ^2), and so

$$\begin{aligned} \|f\|_{L^p(\mathbb{R}^d)} &\leq \|h\|_{L^p(\mathbb{R}^d)} + \varepsilon \stackrel{(\#) \text{ for } h}{\leq} B_{p',d} \|S_{\mathcal{D}}h\|_{L^p(\mathbb{R}^d)} + \varepsilon \\ &\leq \varepsilon + B_{p',d} \|S_{\mathcal{D}}f\|_{L^p(\mathbb{R}^d)} + \underbrace{B_{p',d} \|S_{\mathcal{D}}(f - h)\|_{L^p(\mathbb{R}^d)}}_{\leq B_{p,d} \varepsilon} \\ &\leq B_{p',d} \|S_{\mathcal{D}}f\|_{L^p(\mathbb{R}^d)} + \varepsilon(1 + B_{p',d} B_{p,d}). \end{aligned}$$

Since ε is arbitrary and small we conclude

$$\|f\|_{L^p(\mathbb{R}^d)} \leq B_{p',d} \|S_{\mathcal{D}}f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in L^p(\mathbb{R}^d),$$

which is the lower bound in $(*)$ with $A_{p,d} = \frac{1}{B_{p',d}}$. ■

Summary of the proof. The L^2 case is Pythagoras. For $p \leq 2$ the random signs make $T_{\omega,N}$ a Calderón–Zygmund operator whose bad part never leaves the stopping cubes, giving weak $(1,1)$ and then L^p by interpolation. For $p \geq 2$ nothing new is needed: $T_{\omega,N}$ is self-adjoint, so duality hands the bound over from p' . Khintchine converts a uniform bound over all sign patterns into a bound on the square function, and the lower bound is the upper bound for the dual exponent plus the orthogonality of the $\mathbb{D}_k$.

The Haar system

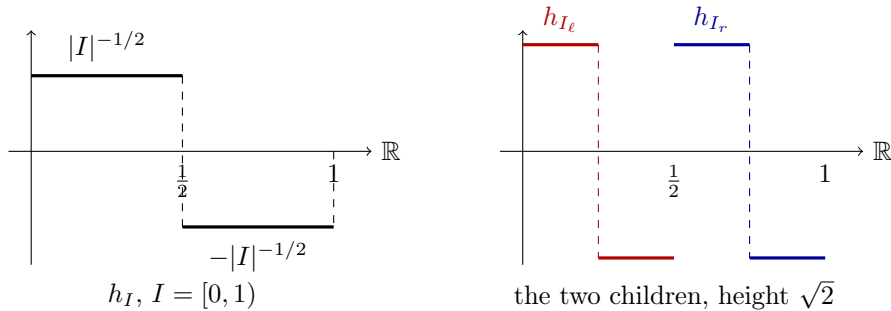
As before let $\mathcal{D}_k$ denote the dyadic intervals of length 2^{-k} in $\mathbb{R}$ and $\mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k$.

Definition. Let $I \in \mathcal{D}$ and let I_ℓ, I_r denote the left and right half of I respectively. Then

$$h_I := \frac{1}{|I|^{1/2}} (\chi_{I_\ell} - \chi_{I_r})$$

is the *Haar function* associated to I .

Remark. h_I and h_J are orthogonal for $I \neq J$, that is $\langle h_I, h_J \rangle = 0$, and $\|h_J\|_{L^2} = 1$ for all $J \in \mathcal{D}$. Indeed, for $I \neq J$ either $I \cap J = \emptyset$, or $I \subsetneq J$, or $J \subsetneq I$. In the first case the supports are disjoint. If $I \subsetneq J$ then I lies inside J_ℓ or inside J_r , where h_J is constant, and $\int h_I = 0$, so $\langle h_I, h_J \rangle = \pm |J|^{-1/2} \int h_I = 0$. The normalisation is $\|h_J\|_{L^2}^2 = |J|^{-1}(|J_\ell| + |J_r|) = 1$.



Each Haar function is a mean-zero step on its interval, and the Haar functions of the children of I live where h_I is constant. That is the whole orthogonality.

Proposition. Let $f \in L^1_{\text{loc}}(\mathbb{R})$. Then for each $k \in \mathbb{Z}$

$$\mathbb{D}_k f = \mathbb{E}_k f - \mathbb{E}_{k-1} f = \sum_{I \in \mathcal{D}_{k-1}} \langle f, h_I \rangle h_I.$$

Proof. Let $I \in \mathcal{D}_{k-1}$, $I = I_\ell \cup I_r$ with $I_\ell, I_r \in \mathcal{D}_k$, and without loss of generality $x \in I_\ell$ (the case $x \in I_r$ is the same with both signs flipped). Then, using $\langle f \rangle_I = \frac{1}{2}(\langle f \rangle_{I_\ell} + \langle f \rangle_{I_r})$,

$$\mathbb{D}_k f(x) = \mathbb{E}_k f(x) - \mathbb{E}_{k-1} f(x) = \langle f \rangle_{I_\ell} - \langle f \rangle_I = \langle f \rangle_{I_\ell} - \frac{1}{2}(\langle f \rangle_{I_\ell} + \langle f \rangle_{I_r}) = \frac{1}{2}(\langle f \rangle_{I_\ell} - \langle f \rangle_{I_r}),$$

while, since $|I_\ell| = |I_r| = \frac{1}{2}|I|$,

$$h_I(x) \langle f, h_I \rangle = \frac{1}{|I|^{1/2}} \cdot \frac{1}{|I|^{1/2}} (|I_\ell| \langle f \rangle_{I_\ell} - |I_r| \langle f \rangle_{I_r}) = \frac{1}{2} (\langle f \rangle_{I_\ell} - \langle f \rangle_{I_r}).$$

The two agree, and for $x \in I$ only the term I of the sum is non-zero. ■

So $\mathbb{D}_k$ is the orthogonal projection onto $\text{span}\{h_I : I \in \mathcal{D}_{k-1}\}$, and in $d = 1$ the function $\chi_Q - \frac{1}{2}\chi_R$ from Lecture 14 is $\frac{|R|^{1/2}}{2}h_R$.

We can write the dyadic square function $S_{\mathcal{D}}$ in terms of Haar coefficients: for fixed x and k exactly one $I \in \mathcal{D}_{k-1}$ contains x , and $|h_I(x)|^2 = |I|^{-1}\chi_I(x)$, so

$$S_{\mathcal{D}} f(x) = \left(\sum_{k \in \mathbb{Z}} |\mathbb{D}_k f(x)|^2 \right)^{1/2} = \left(\sum_{I \in \mathcal{D}} \frac{1}{|I|} \chi_I(x) |\langle f, h_I \rangle|^2 \right)^{1/2}.$$

The theorem above then reads: for $1 < p < \infty$, the L^p norm of f is equivalent to the L^p norm of this square function of its Haar coefficients. For $p = 2$ it is exactly Parseval for the orthonormal basis $\{h_I\}_{I \in \mathcal{D}}$ of $L^2(\mathbb{R})$.

Outlook: multiresolution analysis and wavelet theory. The spaces $V_k = \overline{\text{span}\{\chi_I : I \in \mathcal{D}_k\}}$ are nested, $V_{k-1} \subseteq V_k$, with $\bigcap_k V_k = \{0\}$ and $\bigcup_k V_k = L^2(\mathbb{R})$, and

$$W_k := V_k \ominus V_{k-1} = \text{span}\{h_I : I \in \mathcal{D}_{k-1}\}$$

are the *wavelet spaces*, so that $L^2(\mathbb{R}) = \bigoplus_{k \in \mathbb{Z}} W_k$ and $\mathbb{D}_k$ is the projection onto W_k . This structure, a multiresolution analysis, is the starting point of wavelet theory: one replaces the step function $\chi_{[0,1]}$ and the Haar function by smoother generators (the Meyer wavelet, Daubechies wavelets) whose dilates and translates still form an orthonormal basis with the same nested structure. The Haar system is the simplest wavelet basis, and the dyadic square function is its Littlewood–Paley square function.

End of the course. The dyadic world built in the last four lectures mirrors the continuous one lecture by lecture: $M_{\mathcal{D}}$ for the Hardy–Littlewood maximal function, $S_{\mathcal{D}}$ for the Littlewood–Paley square function, $H_{\mathcal{D}}^1$ and $\text{BMO}_{\mathcal{D}}$ for H^1 and BMO , Haar functions for wavelets, martingale differences for frequency bands. The proofs are the same Calderón–Zygmund arguments, but with no tails: a martingale difference of a function supported in a dyadic cube stays in that cube. That is why the dyadic theory is the model case, and why so much of modern harmonic analysis first proves a statement for dyadic cubes and then transfers it.