

Harmonic Analysis

Lecture 3

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Exercise Classes: Thursdays 13.00–15.00

Theorem: Lebesgue differentiation Thm

Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then

$$\lim_{r \rightarrow 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) dy = f(x) \quad \text{for a.e. } x \in \mathbb{R}^d.$$

Proof: in Seminar 1. (Boundedness of Maximal function, weak $(1, 1)$.) ■

Corollary. For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, one has $|f(x)| \leq Mf(x)$ for a.e. $x \in \mathbb{R}^d$, hence for each $p \in (1, \infty]$, there exists $C_{p,d} > 0$ s.t

$$\|f\|_{L^p(\mathbb{R}^d)} \leq \|Mf\|_{L^p(\mathbb{R}^d)} \leq C_{p,d} \|f\|_{L^p(\mathbb{R}^d)}.$$

Proof: combined results. In detail: by the Lebesgue differentiation theorem there is a null set N such that for every $x \notin N$,

$$\begin{aligned} |f(x)| &= \lim_{r \rightarrow 0} \left| \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) dy \right| \leq \liminf_{r \rightarrow 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy \\ &\leq \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy = M_c f(x) \leq Mf(x), \end{aligned}$$

using $M_c f \leq Mf$ from Lecture 1. This is the first claim.

Now let $p \in (1, \infty]$ and $f \in L^p(\mathbb{R}^d)$. Then $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ (Hölder on each compact K : $\int_K |f| \leq |K|^{1-1/p} \|f\|_{L^p}$), so the pointwise bound applies, and monotonicity of the L^p norm gives $\|f\|_{L^p} \leq \|Mf\|_{L^p}$. For the second inequality: if $p < \infty$, the Corollary of Lecture 2 (Marcinkiewicz applied to weak type $(1, 1)$ and strong type (∞, ∞) of M) gives $\|Mf\|_{L^p} \leq C_{p,d} \|f\|_{L^p}$; if $p = \infty$, Remark 1 of Lecture 1 gives $\|Mf\|_{L^\infty} \leq \|f\|_{L^\infty}$, so $C_{\infty,d} = 1$. ■

Approximate Identity

Def A family $(k_t)_{t>0}$ of functions in $L^1(\mathbb{R}^d)$ is said to be an approximate identity if there exists $M > 0$:

(a) $\int_{\mathbb{R}^d} k_t(x) dx = 1$ for all $t > 0$

(b) $\int_{\mathbb{R}^d} |k_t(x)| dx \leq M$ for all $t > 0$

(c) For every $\delta > 0$, $\lim_{t \rightarrow 0^+} \int_{|x| > \delta} |k_t(x)| dx = 0$.

Example: Let $k \in L^1(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} k(x) dx = 1$. Then $(k_t)_{t>0}$ is an app. identity, where

$$k_t(x) = t^{-d} k(t^{-1}x).$$

Proof: **Exercise**. ■

Theorem (Approximate Identities) Let $(k_t)_{t>0}$ be an app. id.

(i) if $f \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$, then $\|k_t * f - f\|_{L^p(\mathbb{R}^d)} \xrightarrow{t \rightarrow 0^+} 0$.

(ii) if $f \in C_0(\mathbb{R}^d)$, then $k_t * f \xrightarrow{t \rightarrow 0^+} f$ uniformly on $\mathbb{R}^d$.

Proof: (ii) **as in Fourier Analysis, Exercise**.

(i) pick $f \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$. Write

$$k_t * f(x) - f(x) = \int_{\mathbb{R}^d} k_t(y) (f(x-y) - f(x)) dy$$

we consider

Exercise 2 For $y \in \mathbb{R}^d$, let $(\tau_y f)(x) = f(x-y)$. Show that for $f \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$,

$$\|\tau_y f - f\|_{L^p(\mathbb{R}^d)} \xrightarrow{y \rightarrow 0} 0. \quad (\text{need to use LDT})$$

Allows to control the part of the integral where y is small.

Let $\varepsilon > 0$. Find by **(Ex 2)** $\delta > 0$ s.t for $|y| < \delta$, $\|\tau_y f - f\|_{L^p} < \varepsilon$.

Now find $t_0 > 0$ s.th.

$$\int_{|y| > \delta} |k_t(y)| dy < \varepsilon \quad \text{for all } 0 < t \leq t_0.$$

(why does it exist: by condition (c) of the definition, for this fixed δ we have $\int_{|y| > \delta} |k_t(y)| dy \rightarrow 0$ as $t \rightarrow 0^+$, so by definition of the limit there is $t_0 > 0$ with the integral $< \varepsilon$ for all $0 < t \leq t_0$.)

Now for $t \in (0, t_0]$,

$$\begin{aligned} \|k_t * f - f\|_{L^p(\mathbb{R}^d)} &\stackrel{\substack{\leq \\ \uparrow \\ \text{triang. ineq}}}{\leq} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |k_t(y)| |f(x-y) - f(x)| dy \right)^p dx \right)^{1/p} \\ &\stackrel{\substack{(\text{Ex3}) \\ \uparrow \\ \text{Minkowski ineq}}}{\leq} \int_{\mathbb{R}^d} |k_t(y)| \underbrace{\left(\int_{\mathbb{R}^d} |f(x-y) - f(x)|^p dx \right)^{1/p}}_{\|\tau_y f - f\|_{L^p(\mathbb{R}^d)}} dy \quad (\dagger) \end{aligned}$$

Exercise 3 (Minkowski's integral inequality) For $1 \leq p < \infty$ and $F \geq 0$ measurable on $\mathbb{R}^d \times \mathbb{R}^d$,

$$\left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} F(x, y) dy \right)^p dx \right)^{1/p} \leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} F(x, y)^p dx \right)^{1/p} dy.$$

Applied above with $F(x, y) = |k_t(y)| |f(x-y) - f(x)|$.

Now we can split as in continuous case since we control $\|\tau_y f - f\|_{L^p(\mathbb{R}^d)}$ by (Ex 2).

$$(\dagger) \leq \int_{|y| \leq \delta} |k_t(y)| \underbrace{\|\tau_y f - f\|_{L^p}}_{< \varepsilon} dy + \int_{|y| > \delta} |k_t(y)| \underbrace{\|\tau_y f - f\|_{L^p}}_{\leq 2\|f\|_{L^p}} dy$$

first term $\leq M \cdot \varepsilon$ by cond. (b) in def.

second term: $\|\tau_y f - f\|_{L^p} \leq 2\|f\|_{L^p}$ since τ is an isometry on L^p , and the whole term is $\leq 2\|f\|_{L^p} \cdot \varepsilon$ from the t_0 choice.

$$\leq (M + 2\|f\|_{L^p}) \varepsilon$$

Hence we have shown that $\|f * k_t - f\|_{L^p} \xrightarrow{t \rightarrow 0^+} 0$. ■

Examples

1) Let

$$P(x) = \frac{C_d}{(1 + |x|^2)^{\frac{d+1}{2}}}, \quad x \in \mathbb{R}^d$$

where C_d is chosen s.t. $\int P(x) dx = 1$ $\left(C_d = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \right)$, so

$$P_t(x) = t^{-d} P(t^{-1}x) = t^{-d} \frac{C_d}{(1 + (t^{-1}x)^2)^{\frac{d+1}{2}}} = \frac{C_d t}{(t^2 + |x|^2)^{\frac{d+1}{2}}}.$$

Seen $d = 1$ and 2 and general version in PDEs.

2) Let $k = V_d^{-1} \chi_{B(0,1)}$, $V_d = |B(0,1)|$. Then $(k_t)_{t>0}$ is an app. id., and

$$\begin{aligned} M_c f(x) &= \sup_{t>0} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y)| dy = \sup_{t>0} \frac{1}{V_d t^d} \int_{B(0,t)} |f(x-z)| dz \\ &= \sup_{t>0} V_d^{-1} t^{-d} \int_{\mathbb{R}^d} \chi_{B(0,1)}(t^{-1}z) |f(x-z)| dz = (k_t * |f|)(x). \end{aligned}$$

Theorem Let g be a non-neg on $[0, \infty)$, g has at most finitely many discontinuities, g decreasing.

Consider k given by $k(x) = g(|x|)$, $x \in \mathbb{R}^d$. If $k \in L^1(\mathbb{R}^d)$, then for every $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, one has

$$\sup_{t>0} (|f| * k_t)(x) \leq \|k\|_{L^1(\mathbb{R}^d)} M_c f(x) \quad (\text{a.e. } x \in \mathbb{R}^d).$$

Proof: Grafakos $\mathcal{CF}$ 91–92. ■

Corollary. Let g be as in the previous theorem, $k \in L^1(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} k(x) dx = 1$, $|k(x)| \leq g(|x|)$ for all $x \in \mathbb{R}^d$. Then for all $p \in [1, \infty)$, $f \in L^p(\mathbb{R}^d)$,

$$(k_t * f)(x) \xrightarrow{t \rightarrow 0^+} f(x) \quad \text{for a.e. } x \in \mathbb{R}^d.$$

Proof: **Exercise.**

“Generalized Lebesgue Differentiation Theorem”.

“it says we can average over more structures rather than just balls”.

The Schwartz Class and tempered distributions

Def The Schwartz class $\mathcal{S}$ on $\mathbb{R}^d$ is the class of functions in $C^\infty(\mathbb{R}^d)$ with

$$p_{a,b}f = \sup_{x \in \mathbb{R}^d} |x^a \partial^b f(x)| < \infty$$

for each $a \in \mathbb{N}_0^d$, $b \in \mathbb{N}_0^d$. $a = (a_1, \dots, a_d)$, $b = (b_1, \dots, b_d)$, these are multiindices.

Let $\{p_j\}$ denote an enumeration of $\{p_{a,b} : a, b \in \mathbb{N}_0^d\}$. Define $d : \mathcal{S} \times \mathcal{S} \rightarrow [0, \infty)$.

$$d(f, g) = \sum_{j=1}^{\infty} 2^{-j} \underbrace{\frac{p_j(f-g)}{1+p_j(f-g)}}_{\leq 1} \quad \text{so of course finite.}$$

Then d is a metric and $(\mathcal{S}(\mathbb{R}^d), d)$ is complete. (It is a Fréchet space).

One can show $f_n \rightarrow f$ wrt $d \iff p_j(f_n - f) \rightarrow 0 \forall j \in \mathbb{N}$.

Proof: This was seen and done in Topology.

(functional)
A linear map $u : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C}$ is called bounded, if $u(\varphi_k) \xrightarrow[k \rightarrow \infty]{} 0$ whenever $\varphi_k \rightarrow 0$ in $(\mathcal{S}, d)$.

(enough to check at 0 for continuity because linear map, so we can translate back to zero...)

$\implies$ (bounded, if it is continuous at 0).

Def: The space of bounded linear functionals on $\mathcal{S}(\mathbb{R}^d)$ is called the space of tempered distributions and denoted by $\mathcal{S}'(\mathbb{R}^d)$.

Theorem A linear map $u : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C}$ belongs to $\mathcal{S}'(\mathbb{R}^d)$, if and only if there exists $C > 0$, $k, \ell \in \mathbb{N}_0$ such that

$$|u(\varphi)| \leq C \sum_{\substack{|a| \leq k \\ |b| \leq \ell}} p_{a,b}(\varphi)$$

Lit: Grafakos p. 121.

(depends which $p_{a,b}$ I take on the u I took.)

Example $\delta_0 \in \mathcal{S}'(\mathbb{R}^d)$, $\delta_0(f) = f(0)$.