Harmonic Analysis

Lecture 4

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Exercise Classes: Thursdays 13.00–15.00

Schwartz class and tempered distributions

$$\mathscr{S}(\mathbb{R}^d) = \Big\{ \varphi \in C^\infty(\mathbb{R}^d) \ : \ p_{a,b}(\varphi) = \sup_{x\in\mathbb{R}^d} \big| x^a \partial^b \varphi(x) \big| < \infty \ \text{ for all } a,b \in \mathbb{N}_0^d \Big\}$$

$$\mathscr{S}(\mathbb{R}^d)' = \big\{ u : \mathscr{S}(\mathbb{R}^d) \to \mathbb{C} \ : \ u \text{ bounded, linear} \big\}$$

i.e. u linear, $u(\varphi_k) \xrightarrow[k\to\infty]{} 0$ whenever $\varphi_k \xrightarrow[k\to\infty]{} 0$ in $\big(\mathscr{S}(\mathbb{R}^d), d\big)$ *(our topology)*,

$$d(f,g) = \sum_{j=1}^{\infty} 2^{-j} \frac{p_j(f-g)}{1+p_j(f-g)}, \qquad \{p_j\}_{j=1}^\infty \text{ enumeration of } \{p_{a,b}\}_{a,b\in\mathbb{N}_0^d}.$$

$\mathscr{S}(\mathbb{R}^d)'$ is the space of tempered distributions.

Write

$$u(\varphi) = \langle u, \varphi \rangle = \text{“} \int_{\mathbb{R}^d} u(t)\varphi(t)\, dt \text{”}$$

“ ” because u is not necessarily a function.

examples

1) $u = \delta_0$

2) $f \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, define

$$\langle u_f, \varphi \rangle = \int_{\mathbb{R}^d} \underset{\substack{\uparrow \\ L^p}}{f(t)} \ \underset{\substack{\uparrow \\ \mathscr{S}}}{\varphi(t)}\, dt$$

3) For $\varphi \in \mathscr{S}(\mathbb{R})$, define p.v.$\left(\frac{1}{x}\right) : \mathscr{S}(\mathbb{R}) \to \mathbb{C}$,

$$\left\langle \text{p.v.}\left(\tfrac{1}{x}\right), \varphi \right\rangle = \text{“} \int \frac{\varphi(x)}{x}\, dx \text{”}$$

$\uparrow$ *not integrable, but we can take an appropriate limit.*

$$\left\langle \text{p.v.}\left(\tfrac{1}{x}\right), \varphi \right\rangle = \lim_{\varepsilon\to 0^+} \int_{\varepsilon \le |x| \le 1} \frac{\varphi(x)}{x}\, dx + \underbrace{\int_{|x|>1} \frac{\varphi(x)}{x}\, dx}_{\text{well defined: } \frac{1}{x}\chi_{\mathbb{R}\setminus[-1,1]} \in L^2}$$

exercise question for next week.

1. Differentiation on $\mathscr{S}(\mathbb{R}^d)'$

Recall that for any $\varphi, \psi \in \mathscr{S}(\mathbb{R}^d)$, then $a \in \mathbb{N}_0^d$

$$\int_{\mathbb{R}^d} \partial^a \varphi(x)\, \psi(x)\, dx = (-1)^{|a|} \int_{\mathbb{R}^d} \varphi(x)\, \partial^a \psi(x)\, dx$$

define for $u \in \mathscr{S}(\mathbb{R}^d)'$,

$$\langle \partial^a u, \varphi \rangle := (-1)^{|a|} \langle u, \partial^a \varphi \rangle \qquad (\varphi \in \mathscr{S}(\mathbb{R}^d))$$

"outrageous things done on u can be thrown on φ"

2. Convolution with a Schwartz function on $\mathscr{S}(\mathbb{R}^d)'$

For $\eta \in \mathscr{S}(\mathbb{R}^d)$, we define $\widetilde{\eta}(x) = \eta(-x)$.

Recall: For $\eta, \varphi, \psi \in \mathscr{S}(\mathbb{R}^d)$,

$$\langle \eta * \psi, \varphi \rangle = \langle \psi, \varphi * \widetilde{\eta} \rangle.$$

For $u \in \mathscr{S}(\mathbb{R}^d)'$, $\eta \in \mathscr{S}(\mathbb{R}^d)$, define $u * \eta$ by

$$\langle u * \eta, \varphi \rangle := \langle u, \varphi * \widetilde{\eta} \rangle, \qquad \varphi \in \mathscr{S}(\mathbb{R}^d).$$

Theorem: If $u \in \mathscr{S}(\mathbb{R}^d)'$, $\eta \in \mathscr{S}(\mathbb{R}^d)$, then $u * \eta$ can be identified with a C^∞ function.

Definition. The Hilbert transform, H is defined on $\mathscr{S}(\mathbb{R})$ by

$$(H\varphi)(y) := \Big(\tfrac{1}{\pi}\, \text{p.v.}\big(\tfrac{1}{x}\big) * \varphi \Big)(y).$$

3. Fourier transform

Recall that for $f \in \mathscr{S}(\mathbb{R}^d)$ or even $f \in L^1(\mathbb{R}^d)$

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i\, x\cdot\xi}\, dx \qquad \text{and inverse} \qquad \check{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{2\pi i\, x\cdot\xi}\, dx$$

(note $x \cdot \xi = \sum_{i=1}^d x_i \xi_i$)

Recall that $\check{\widehat{f}} = f$ for $f \in \mathscr{S}(\mathbb{R}^d)$. We define for $u \in \mathscr{S}(\mathbb{R}^d)'$:

$$\langle \widehat{u}, \varphi \rangle := \langle u, \widehat{\varphi} \rangle, \qquad \varphi \in \mathscr{S}(\mathbb{R}^d).$$

For this to be well defined we need a very important property $\mathcal{F} : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$, also define inverse

$$\langle \check{u}, \varphi \rangle := \langle u, \check{\varphi} \rangle \qquad (\varphi \in \mathscr{S}(\mathbb{R}^d))$$

Proposition 10 Let $k \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} k(x)\, dx = 1$. With $k_t(x) := t^{-d} k(t^{-1}x)$, this gives an app. id. $(k_t)_{t>0}$. Then $k_t \xrightarrow[t\to 0]{} \delta_0$ in $\mathscr{S}(\mathbb{R}^d)'$, i.e.

$$\langle k_t, \varphi \rangle \xrightarrow[t\to 0]{} \langle \delta_0, \varphi \rangle = \varphi(0), \qquad \forall\, \varphi \in \mathscr{S}(\mathbb{R}^d).$$

Proof. Fix $\varphi \in \mathscr{S}(\mathbb{R}^d)$, $t > 0$. Then

$$\langle k_t, \varphi\rangle = \int_{\mathbb{R}^d} t^{-d} k(t^{-1}x)\varphi(x)\,dx = \int_{\mathbb{R}^d} k(y) \underbrace{\varphi(ty)}_{\substack{\downarrow\, t\to 0 \\ \varphi(0) \text{ for each } y\in\mathbb{R}^d}} dy \xrightarrow[t\to 0]{\text{DCT}} \int_{\mathbb{R}^d} k(y)\varphi(0)\,dy = \varphi(0).$$

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Theorem For $u \in \mathscr{S}(\mathbb{R}^d)'$, $\eta \in \mathscr{S}(\mathbb{R}^d)$, $u * \eta \in C^\infty(\mathbb{R}^d)$, with

$$(u*\eta)(x) = \langle u, \underbrace{\tau_x \widetilde{\eta}}_{\in \mathscr{S}(\mathbb{R}^d)} \rangle.$$

Convolution operators

Def A vector space X of mble functions on $\mathbb{R}^d$ is said to be closed under translation, if for every $f \in X$, $h \in \mathbb{R}^d$, then $\tau_h f \in X$, where $\tau_h f(x) = f(x-h)$.

Def. Let X, Y be two spaces of mble functions on $\mathbb{R}^d$ which are closed under translation. Let $T: X \to Y$ be an operator. We say that T is translation-invariant, if $T \circ \tau_h = \tau_h \circ T \quad \forall\, h \in \mathbb{R}^d$.

Theorem Suppose that $T: L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is linear, bounded and translation invariant. Then there exists $u \in \mathscr{S}(\mathbb{R}^d)'$ s.th.

- $Tf = u * f \quad \forall\, f \in \mathscr{S}(\mathbb{R}^d)$
- $\widehat{u}$ is an L^∞ function on $\mathbb{R}^d$.

Proof: Duoandikoetxea.

Theorem Suppose that $u \in \mathscr{S}(\mathbb{R}^d)'$ is s.t:

- $\widehat{u}$ agrees with an L^∞ function on $\mathbb{R}^d$
- u agrees on $\mathbb{R}^d \setminus \{0\}$ with a $L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ function K with

$$\sup_{y\in\mathbb{R}^d} \int_{|x|>2|y|} \big|K(x-y) - K(x)\big|\,dx < \infty$$

(this is a smoothness condition) $\boxed{\text{CZO}}$ *Calderón–Zygmund*

then the operator defined on $\mathscr{S}(\mathbb{R}^d)$ by $T\varphi = u * \varphi$ can be extended to a bounded linear operator $T: L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ for each $1 < p < \infty$.

Example 1) The Hilbert transform H given by $Hf = \frac{1}{\pi}\,\text{p.v.}\frac{1}{x} * f$ for $f \in \mathscr{S}(\mathbb{R})$ extends to a bounded lin op on $L^p(\mathbb{R})$ for each p with $1 < p < \infty$.

2) For $j \in \{1, \ldots, d\}$, the jth Riesz transform R_j is defined by $R_j f = W_j * f$ $(f \in \mathscr{S}(\mathbb{R}^d))$,

$$\langle W_j, \varphi\rangle := \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} \lim_{\varepsilon\to 0^+} \int_{|y|>\varepsilon} \frac{y_j}{|y|^{d+1}}\,\varphi(y)\,dy \qquad (\varphi \in \mathscr{S}(\mathbb{R}^d)).$$

The Calderón–Zygmund decomposition

Def: A set $Q \subseteq \mathbb{R}^d$ is said to be a dyadic cube, if there exist $k, m_1, \ldots, m_d \in \mathbb{Z}$ s.th

$$Q = \big[2^k m_1,\, 2^k(m_1+1)\big) \times \cdots \times \big[2^k m_d,\, 2^k(m_d+1)\big)$$

The set of all dyadic cubes is denoted by $\mathcal{D}_d$.

Exercise For any $Q, Q' \in \mathcal{D}_d$ we have $Q \cap Q' = \emptyset$ or $Q \subseteq Q'$ or $Q' \subseteq Q$.

Theorem (Calderón–Zygmund decomposition)

Let $f \in L^1(\mathbb{R}^d)$, $a > 0$. Then there exists a collection $\mathcal{Q}$ of mutually disjoint dyadic cubes Q in $\mathbb{R}^d$, such that with

$$\underbrace{\mathscr{O} := \bigcup_{Q \in \mathcal{Q}} Q}_{\text{“bad”}}, \qquad \underbrace{F = \mathbb{R}^d \setminus \mathscr{O}}_{\text{“good”}}$$

(a) $|f(x)| \leq a$ for a.e. $x \in F$

(b) $a < \dfrac{1}{|Q|} \displaystyle\int_Q |f(x)|\, dx \leq 2^d a.$

$\mathcal{Q}$ is called the Calderón–Zygmund decomp of f at height a.