Harmonic Analysis
Lecture 5

Teodoras Paura

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Exercise Classes: Thursdays 13.00–15.00

Proof and construction worth keeping in mind.

Theorem 3 (Calderón–Zygmund operators of convolution type)

Suppose that $u \in \mathscr{S}(\mathbb{R}^d)'$ such that

- $\widehat{u}$ agrees with an L^∞-function
- (†) u agrees on $\mathbb{R}^d \setminus \{0\}$ with a $L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ function K such that

$$\sup_{y \in \mathbb{R}^d} \int_{|x|>2|y|} \big|K(x-y) - K(x)\big|\, dx < \infty$$

Then the operator T initially defined on $\mathscr{S}(\mathbb{R}^d)$ by $Tf = u * f$ can be extended to a bounded linear operator on each $L^p(\mathbb{R}^d)$, $1 < p < \infty$.

"agrees with": $\widehat{u}$ agrees with g, $g \in L^\infty(\mathbb{R}^d)$ means:

$$\langle \widehat{u}, \varphi \rangle = \langle g, \varphi \rangle \qquad \forall\, \varphi \in \mathscr{S}(\mathbb{R}^d).$$

(†) is an extra condition to get away from L^2. "Hörmander condition".

Theorem 1 (Calderón–Zygmund decomposition, cube version) Let $f \in L^1(\mathbb{R}^d)$, $a > 0$. Then there exists a collection of mutually disjoint dyadic cubes $\mathcal{Q}$ in $\mathbb{R}^d$ such that for $\mathscr{O} = \bigcup_{Q \in \mathcal{Q}} Q$ and $F = \mathbb{R}^d \setminus \mathscr{O}$, then

(a) $|f(x)| \le a$ for a.e. $x \in F$

(b) $a < \dfrac{1}{|Q|} \displaystyle\int_Q |f(x)|\, dx \le 2^d a \quad \forall\, Q \in \mathcal{Q}$.

Proof: If $|f(x)| \le a$ a.e. $x \in \mathbb{R}^d$, set $\mathcal{Q} = \emptyset$, $F = \mathbb{R}^d$.

In the general case, choose $N \in \mathbb{N}$ such that

$$\frac{1}{2^{Nd}} \underbrace{\int_{\mathbb{R}^d} |f(x)|\, dx}_{\|f\|_{L^1} < \infty} \;\le\; a.$$

Now decompose $\mathbb{R}^d$ into a mesh of mutually disjoint dyadic cubes of side length 2^N. Call this collection $\mathcal{Q}_0$.

Pick $Q \in \mathcal{Q}_0$. Bisect Q in each coordinate direction to obtain 2^d mutually disjoint dyadic cubes of side length 2^{N-1}. Let $Q' \subseteq Q$ one of these.

Case 1: $\dfrac{1}{|Q'|}\displaystyle\int_{Q'}|f(x)|\,dx > a$

Case 2: $\dfrac{1}{|Q'|}\displaystyle\int_{Q'}|f(x)|\,dx \le a.$

In case 1, we stop and let $Q' \in \mathcal{Q}$. Indeed, note that

$$a < \frac{1}{|Q'|}\int_{Q'}|f(x)|\,dx \;\le\; \frac{2^d}{|Q|}\int_{Q}|f(x)|\,dx \;=\; 2^d\,\frac{1}{2^{Nd}}\int_{Q}|f(x)|\,dx \;\le\; 2^d a.$$

Hence Q' satisfies (b).

For cubes in case 2, we continue the process of subdividing them until we get, if ever, cubes of case 1. We denote the collection of all case 1-cubes we obtain by $\mathcal{Q}$.

Now check if all the conditions are met.

The cubes are mutually disjoint by construction.

Let $Q \in \mathcal{Q}$. Then by definition of the collection, $\widehat{Q}$ (the dyadic parent of Q) satisfies case 2. Hence

$$a < \frac{1}{|Q|}\int_{Q}|f(x)|\,dx \;\le\; \frac{2^d}{|\widehat{Q}|}\int_{\substack{\widehat{Q}\\ \text{dyadic parent}}}|f(x)|\,dx \;\le\; 2^d a.$$

So we have taken care of part (b).

For $x \in F = \mathbb{R}^d \setminus \bigcup_{Q\in\mathcal{Q}} Q$, let (Q_n) be the sequence of dyadic cubes of side length 2^n, $x \in Q_n$. Each one of these Q_n cubes are going to be a case 2 cube, otherwise we would have stopped (case 1) and this x would not be in F.

Lebesgue Diff Theorem $\Rightarrow$

$$|f(x)| = \lim_{n\to-\infty}\frac{1}{|Q_n|}\int_{Q_n}|f(y)|\,dy \;\underset{\substack{\uparrow\\ \textit{case 2}}}{\le}\; a \qquad \text{holds a.e.}$$

(used to see concentric balls, but works as a variant for dyadic cubes (exercise))

property (a) done. *(Case 1 cubes are the maximal cubes, not all cubes that satisfy, because we could take subcubes.)* ∎

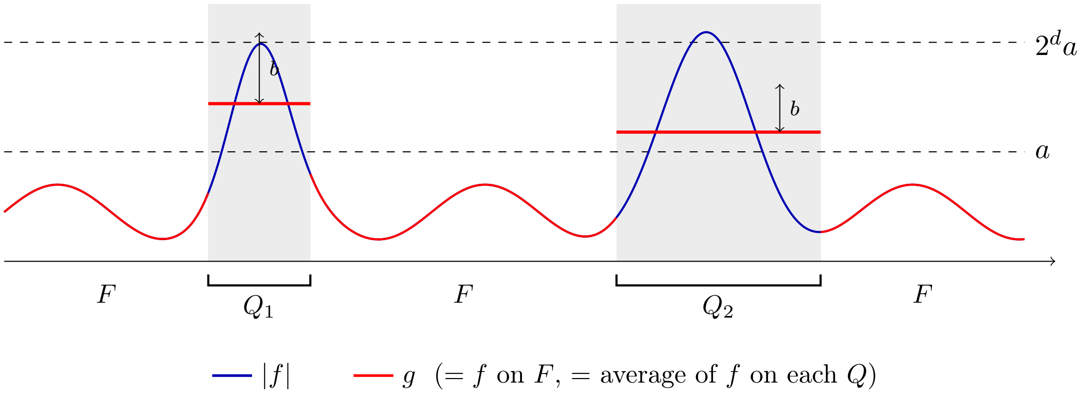
Theorem 2 (Calderón–Zygmund decomposition of f: good and bad part) Suppose that $f \in L^1(\mathbb{R}^d)$, $a > 0$. Then there exists a collection of mutually disjoint dyadic cubes $\mathcal{Q}$ and measurable functions g, b, constants $C_1, C_2 > 0$ such that

(a) $f = g + b$ a.e. on $\mathbb{R}^d$

(b) $|g(x)| \le C_1 a$ for a.e. $x \in \mathbb{R}^d$

(c) $\operatorname{supp}(b) \subseteq \bigcup_{Q\in\mathcal{Q}} Q$, $\displaystyle\int_Q b(x)\,dx = 0 \quad \forall\, Q \in \mathcal{Q}$

(d) $\dfrac{1}{|Q|}\displaystyle\int_Q |b(x)|\,dx \le C_2 a \quad \forall\, Q \in \mathcal{Q}.$



The cubes $Q \in \mathcal{Q}$ (shaded) are exactly where the averages of $|f|$ exceed a; there $a < \frac{1}{|Q|}\int_Q |f| \le 2^d a$, so the flat red level sits between the two dashed lines. Off the cubes $|f| \le a$ and $g = f$. The bad part $b = f - g$ lives only on the cubes and has mean zero on each of them, so g is bounded by $2^d a$ everywhere while all the large oscillation of f has been pushed into b.

Proof. Let $\mathcal{Q}$ be the collection from Theorem 1. We define

$$g(x) = \begin{cases} f(x) & \text{if } x \in F \\ \dfrac{1}{|Q|}\displaystyle\int_Q f(y)\,dy & \text{if } x \in Q \text{ and } Q \in \mathcal{Q} \end{cases}$$

$$b(x) = f(x) - g(x)$$

(a) satisfied by definition of b.

(b) For a.e. $x \in F$, $|g(x)| \le a$. For $x \in Q$, $Q \in \mathcal{Q}$,

$$|g(x)| \;\le\; \frac{1}{|Q|}\int_Q |f(x)|\,dx \;\le\; 2^d a \qquad \text{by (b) in Theorem 1.}$$

So (b) holds with $C_1 = 2^d$.

(c) $\operatorname{supp} b \subseteq \bigcup_{Q\in\mathcal{Q}} Q$, since $b(x) = 0$ for $x \in \mathbb{R}^d \setminus \bigcup_{Q\in\mathcal{Q}} Q$.

For $Q \in \mathcal{Q}$,

$$\frac{1}{|Q|}\int_Q b(x)\,dx = \frac{1}{|Q|}\int_Q \big(f(x) - g(x)\big)\,dx = \frac{1}{|Q|}\int_Q f(x)\,dx - \frac{1}{|Q|}\int_Q \underbrace{\Big(\frac{1}{|Q|}\int_Q f(y)\,dy\Big)}_{=g \text{ on } Q}\,dx$$

$$= \frac{1}{|Q|}\int_Q f(x)\,dx - \frac{1}{|Q|}\int_Q f(y)\,dy = 0.$$

(d)

$$\frac{1}{|Q|}\int_Q |b(x)|\,dx \;\le\; \underbrace{\frac{1}{|Q|}\int_Q |f(x)|\,dx}_{\le 2^d a} + \underbrace{\frac{1}{|Q|}\int_Q |g(x)|\,dx}_{\le \frac{1}{|Q|}\int_Q |f(x)|\,dx \le 2^d a} \;\le\; 2^{d+1} a$$

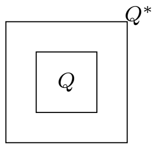
$C_2 = 2^{d+1}$. Thus holds. ∎

Proof of Theorem 3:

Fact 1 ($p=2$). We already know that T extends to a bounded linear operator on $L^2(\mathbb{R}^d)$, since

$$\|u*\varphi\|_{L^2} = \|\mathcal{F}(u*\varphi)\|_{L^2} = \|\widehat{u}\cdot\widehat{\varphi}\|_{L^2} \le \|\widehat{u}\|_{L^\infty}\|\widehat{\varphi}\|_{L^2} = \|\widehat{u}\|_{L^\infty}\cdot\|\varphi\|_{L^2} \qquad \text{for } \varphi\in\mathscr{S}(\mathbb{R}^d).$$

Fact 2 *(For $Q\in\mathcal{Q}$ (or just for any cube), let Q^* be the cube with twice the side length of Q and the same centre. (not necessarily a dyadic))*



Let Q be a cube in $\mathbb{R}^d$. If $b_Q\in L^\infty(\mathbb{R}^d)$, $\operatorname{supp} b_Q\subseteq Q$, for a.e. $x\in\mathbb{R}^d\setminus Q^*$,

$$Tb_Q(x) = \int_Q K(x-y)\,b_Q(y)\,dy.$$

Idea: Show weak $(1,1)$ boundedness and then use Marcinkiewicz interpolation between $p=1$ and $p=2$.

Let $f\in\mathscr{S}(\mathbb{R}^d)$ and $a>0$. We want to show

$$\big|\{|Tf(x)|>a\}\big| \;\le\; \frac{C}{a}\int_{\mathbb{R}^d}|f(x)|\,dx,$$

where C is an absolute constant.

Using Theorem 2 and write $f=g+b$, for this given a. Hence $|g(x)|\le C_1 a$ by property (b) of Theorem 2.

$$\begin{aligned}
\int_{\mathbb{R}^d}|g(x)|^2\,dx \;&\le\; 2^d a\int_{\mathbb{R}^d}|g(x)|\,dx \;=\; 2^d a\Big(\int_F|g(x)|\,dx+\int_{\mathscr{O}}|g(x)|\,dx\Big)\\
&\le\; 2^d a\Big(\int_F|f(x)|\,dx+\sum_{Q\in\mathcal{Q}}\int_Q\Big|\frac{1}{|Q|}\int_Q f(y)\,dy\Big|\,dx\Big)\\
&\le\; 2^d a\Big(\int_F|f(x)|\,dx+\sum_{Q\in\mathcal{Q}}\int_Q|f(x)|\,dx\Big) \;=\; 2^d a\,\|f\|_{L^1}
\end{aligned}$$

$$\Longrightarrow\quad g\in L^2(\mathbb{R}^d),\qquad \|g\|_{L^2}^2\le 2^d a\,\|f\|_{L^1}$$

$$\Longrightarrow\quad b=\underset{\substack{\uparrow\\ \mathscr{S}(\mathbb{R}^d)}}{f}-g\in L^2(\mathbb{R}^d)$$

Hence Tb is well defined and we may write:

$$Tf = Tg+Tb,$$

hence we know we may estimate

$$\big|\{x\in\mathbb{R}^d:|Tf(x)|>a\}\big| \;\le\; \big|\{x\in\mathbb{R}^d:|Tg(x)|>\tfrac{a}{2}\}\big|+\big|\{x\in\mathbb{R}^d:|Tb(x)|>\tfrac{a}{2}\}\big|$$

and

$$\left|\{x \in \mathbb{R}^d : |Tg(x)| > \tfrac{a}{2}\}\right| \le \frac{1}{(\frac{a}{2})^2} \int_{\mathbb{R}^d} |Tg(x)|^2 \, dx$$

$$\underset{\substack{\uparrow \\ \textit{Fact 1}}}{\le} \frac{4}{a^2} \|\widehat{u}\|_{L^\infty}^2 \underbrace{\|g\|_{L^2}^2}_{\le 2^d a \|f\|_{L^1}} \le \frac{2^{d+2}}{a} \|\widehat{u}\|_{L^\infty}^2 \|f\|_{L^1}.$$

For the second bound, we will do on Friday, we shall use the Hörmander condition.