Harmonic Analysis

Lecture 6

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Exercise Classes: Thursdays 13.00–15.00

(I) $u \in \mathscr{S}(\mathbb{R}^d)'$, $\widehat{u}$ agrees with an L^∞ function on $\mathbb{R}^d$

(II) there exists $K \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ such that

$$\langle u, \varphi \rangle = \int_{\mathbb{R}^d} K(y)\, \overline{\varphi}(y)\, dy \qquad \text{for } \varphi \in \mathscr{S}(\mathbb{R}^d),\ \operatorname{supp} \varphi \subseteq \mathbb{R}^d \setminus \{0\}$$

$$(Tf = u * f \text{ for } f \in \mathscr{S}(\mathbb{R}^d)), \quad A = \sup_y \int_{|x|>2|y|} \big|K(x-y) - K(x)\big|\, dx < \infty.$$

<u>Show:</u> $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ bounded for $1 < p < \infty$ *(strong type (p,p))*.

<u>Show:</u> $\big|\{|Tf(x)| > a\}\big| \le \dfrac{C}{a} \|f\|_{L^1}$ for $a > 0$.

Calderón–Zygmund decomposition $f = b + g$, dyadic cubes collection $\mathcal{Q}$, for which we have a bunch of properties.

$$\big|\{|Tf(x)| > a\}\big| \;\le\; \underbrace{\big|\{|Tg(x)| > \tfrac{a}{2}\}\big|}_{\substack{\textit{this we showed} \\ \textit{is bounded in Lecture 5}}} + \underbrace{\big|\{|Tb(x)| > \tfrac{a}{2}\}\big|}_{\textit{now!}}$$

<u>Recall Fact 2:</u> Let Q be a cube in $\mathbb{R}^d$. If $b_Q \in L^\infty(\mathbb{R}^d)$ and $\operatorname{supp} b_Q \subseteq Q$, then for $x \in (Q^*)^c$ *(enlarged)*,

$$T(b_Q)(x) = \int_Q K(x-y)\, b_Q(y)\, dy.$$

Note that this is not (II) because b_Q is not necessarily a Schwartz function. We would need to show that it is bounded, like using density of Schwartz functions.

$$\big|\{|Tb(x)| > \tfrac{a}{2}\}\big| \;\le\; \Big|\bigcup_{Q\in\mathcal{Q}} Q^*\Big| + \Big|\Big\{x \notin \bigcup_{Q\in\mathcal{Q}} Q^* : |Tb(x)| > \tfrac{a}{2}\Big\}\Big|$$

Note that

$$\Big|\bigcup_{Q\in\mathcal{Q}} Q^*\Big| \;\le\; \sum_{Q\in\mathcal{Q}} |Q^*| \;\le\; \sum_{Q\in\mathcal{Q}} 2^d d^{d/2} |Q| \;\le\; 2^d d^{d/2}\, \frac{\|f\|_{L^1}}{a}$$

(since $\ell(Q^*) = 2\sqrt{d}\,\ell(Q)$; and $\sum_{Q\in\mathcal{Q}} |Q| \le \frac{1}{a} \sum_{Q\in\mathcal{Q}} \int_Q |f| \le \frac{\|f\|_{L^1}}{a}$ by property (b) of Theorem 1, $a|Q| < \int_Q |f|$)

so that term is fine, now the second one.

Exercise: $\left|\Big(T\Big(\sum_{Q\in\mathcal{Q}} b_Q\Big)\Big)(x)\right| \le \sum_{Q\in\mathcal{Q}} \big|(Tb_Q)(x)\big|$ for a.e. $x\in\mathbb{R}^d$. The disjointness will help! Note $b_Q = b\cdot\chi_Q$.

Hence

$$
\begin{aligned}
\left|\Big\{x\in\mathbb{R}^d\setminus\bigcup_{Q\in\mathcal{Q}} Q^* : |Tb(x)|>\tfrac{a}{2}\Big\}\right|
&\underset{\text{Chebyshev}}{\underset{\uparrow}{\le}} \frac{2}{a}\int_{\mathbb{R}^d\setminus\bigcup_{Q\in\mathcal{Q}} Q^*} |Tb(x)|\,dx \\
&\underset{b=\sum_{Q'} b_{Q'}}{\underset{\text{exercise}}{\underset{\uparrow}{\le}}} \frac{2}{a}\sum_{Q'\in\mathcal{Q}}\int_{\mathbb{R}^d\setminus\bigcup_{Q\in\mathcal{Q}} Q^*} |Tb_{Q'}(x)|\,dx \\
&\underset{\mathbb{R}^d\setminus\bigcup_Q Q^*\subseteq\mathbb{R}^d\setminus Q'^*}{\underset{\text{enlarge domain:}}{\underset{\uparrow}{\le}}} \frac{2}{a}\sum_{Q\in\mathcal{Q}}\int_{\mathbb{R}^d\setminus Q^*} |Tb_Q(x)|\,dx \\
&\underset{\text{Fact 2}}{\underset{\uparrow}{=}} \frac{2}{a}\sum_{Q\in\mathcal{Q}}\int_{\mathbb{R}^d\setminus Q^*}\left|\int_Q K(x-y)\,b_Q(y)\,dy\right| dx \\
&= \frac{2}{a}\sum_{Q\in\mathcal{Q}}\int_{\mathbb{R}^d\setminus Q^*}\left|\int_Q \big(K(x-y)-K(x-c_Q)\big)\,b_Q(y)\,dy\right| dx
\end{aligned}
$$

On the indices: Q' is the cube whose piece $b_{Q'}$ we are looking at, while the domain of integration still has all the Q^* removed. Removing fewer sets gives a bigger domain, and the integrand is nonnegative, so for each fixed Q' the integral over $\mathbb{R}^d\setminus\bigcup_Q Q^*$ is at most the integral over $\mathbb{R}^d\setminus Q'^*$. After that step only one cube is involved per term, so we rename $Q'\to Q$. The enlargement is exactly what Fact 2 needs: it applies on $\mathbb{R}^d\setminus Q^*$ for the cube Q carrying b_Q.

c_Q: centre of Q. The last equality uses $\int_Q b_Q(y)\,dy = 0$, property (c) of Theorem 2, so subtracting $K(x-c_Q)\int_Q b_Q(y)\,dy$ changes nothing.

$$
\begin{aligned}
&\underset{\text{triangle, Tonelli}}{\underset{\uparrow}{\le}} \frac{2}{a}\sum_{Q\in\mathcal{Q}}\int_Q |b_Q(y)| \underbrace{\int_{\mathbb{R}^d\setminus Q^*}\big|K(x-y)-K(x-c_Q)\big|\,dx}_{\le A \text{ Hörmander condition (exercise)}}\,dy \\
&\le \frac{2A}{a}\sum_{Q\in\mathcal{Q}}\int_Q |b_Q(y)|\,dy \underset{\frac{1}{|Q|}\int_Q|b|\le 2^{d+1}a}{\underset{\text{Theorem 2(d), Lecture 5:}}{\underset{\uparrow}{\le}}} \frac{2A}{a}\sum_{Q\in\mathcal{Q}} |Q|\cdot 2^{d+1}a \\
&\underset{|Q|<\frac{1}{a}\int_Q|f|}{\underset{\text{Theorem 1(b), Lecture 5:}}{\underset{\uparrow}{\le}}} \frac{2^{d+2}A}{a}\sum_{Q\in\mathcal{Q}}\int_Q |f(y)|\,dy \\
&\underset{\text{cubes disjoint}}{\underset{\uparrow}{=}} \frac{2^{d+2}A}{a}\int_{\bigcup_{Q\in\mathcal{Q}} Q} |f(y)|\,dy \le \frac{2^{d+2}A}{a}\|f\|_{L^1(\mathbb{R}^d)}
\end{aligned}
$$

Hence, T is weak $(1,1)$ with

$$C = 2^{d+2}\|\widehat{u}\|_{L^\infty}^2 + 2^d d^{d/2} + 2^{d+2}A.$$

For $f \in L^1(\mathbb{R}^d)$, choose (f_n) in $\mathscr{S}(\mathbb{R}^d)$ with $\|f-f_n\|_{L^1} \xrightarrow[n\to\infty]{} 0$. Then (Tf_n) is Cauchy in measure: for each $\varepsilon > 0$, $\delta > 0$, $\exists N$ such that

$$\big|\{|Tf_n(x) - Tf_m(x)| > \varepsilon\}\big| < \delta \qquad \text{for } n, m \geq N.$$

Hence Tf_n converges in measure to a function g. *(Need: g does not depend on the chosen sequence.)*

For any other sequence $\widetilde{f}_n$ with $\|\widetilde{f}_n - f\|_{L^1} \to 0$, $(Tf_n - T\widetilde{f}_n)$ converges to 0 in measure, $T\widetilde{f}_n \to \widetilde{g}$ in measure, so $g = \widetilde{g}$, so define

$$\boxed{Tf = g}$$

Hence T is strong type $(2,2)$ and weak type $(1,1)$. By Marcinkiewicz interpolation, T is strong type (p,p) for $p \in (1,2)$:

$$\Longrightarrow \quad T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \quad \text{bounded for } p \in (1,2),$$

$$\|T\|_{L^p \to L^p} \leq \frac{B}{p-1} \quad \text{as } p \to 1^+, \qquad \text{for } B = B(d, A, \|\widehat{u}\|_{L^\infty}).$$

Now let $p \in (2,\infty)$, $\frac{1}{p'} + \frac{1}{p} = 1$.

Let $f \in \mathscr{S}(\mathbb{R}^d)$. Then

$$\begin{aligned}
\|Tf\|_{L^p} &\underset{\substack{\uparrow \\ \textit{duality + density}}}{=} \sup_{\substack{g \in \mathscr{S}(\mathbb{R}^d) \\ \|g\|_{L^{p'}}=1}} \left|\int_{\mathbb{R}^d} Tf(x)\,\overline{g}(x)\,dx\right| = \sup_{\substack{g \in \mathscr{S} \\ \|g\|_{L^{p'}}=1}} \big|\langle u * f, \overline{g}\rangle\big| \\
&= \sup_{\substack{g \in \mathscr{S}(\mathbb{R}^d) \\ \|g\|_{L^{p'}}=1}} \big|\langle u, \overline{g} * \widetilde{f}\rangle\big| = \sup_{\substack{g \in \mathscr{S} \\ \|g\|_{L^{p'}}=1}} \big|\langle u * \widetilde{\overline{g}}, \widetilde{f}\rangle\big| \qquad \textit{recall } \widetilde{f}(x) = f(-x) \\
&= \sup_{\substack{g \in \mathscr{S} \\ \|g\|_{L^{p'}}=1}} \big|\langle T\widetilde{\overline{g}}, \widetilde{f}\rangle\big| \leq B_{p'}\,\|f\|_{L^p}
\end{aligned}$$

$B_{p'} = \|T\|_{L^{p'} \to L^{p'}}$, $p' \in (1,2]$. (how?) *Hölder:* $|\langle T\widetilde{\overline{g}}, \widetilde{f}\rangle| \leq \|T\widetilde{\overline{g}}\|_{L^{p'}} \|\widetilde{f}\|_{L^p} \leq B_{p'} \|\widetilde{\overline{g}}\|_{L^{p'}} \|f\|_{L^p} = B_{p'}\|f\|_{L^p}$, *since* $\|\widetilde{\overline{g}}\|_{L^{p'}} = \|g\|_{L^{p'}} = 1$.

This holds for every $f \in \mathscr{S}(\mathbb{R}^d)$: $\|Tf\|_{L^p} \leq B_{p'}\|f\|_{L^p}$. Since $\|T\|_{L^p \to L^p} = \sup_{f \neq 0} \|Tf\|_{L^p}/\|f\|_{L^p}$ is the smallest constant C with $\|Tf\|_{L^p} \leq C\|f\|_{L^p}$ for all f (by density of $\mathscr{S}$ the supremum over $\mathscr{S}$ equals the one over L^p), and $B_{p'}$ is such a constant,

$$\|T\|_{L^p \to L^p} \;\leq\; B_{p'} \;=\; \|T\|_{L^{p'} \to L^{p'}} \qquad (p \in (2,\infty)). \tag{$\dagger$}$$

So T is bounded on L^p for $p \in (2,\infty)$ as well.

Why equality. Now that T is bounded on L^p too, run the same argument with the roles of p and p' swapped. For $f \in \mathscr{S}(\mathbb{R}^d)$, dualise against $g \in \mathscr{S}(\mathbb{R}^d)$ with $\|g\|_{L^p} = 1$ and throw T onto g exactly as above:

$$\|Tf\|_{L^{p'}} = \sup_{\substack{g \in \mathscr{S} \\ \|g\|_{L^p}=1}} \big|\langle Tf, \overline{g}\rangle\big| = \sup_{\substack{g \in \mathscr{S} \\ \|g\|_{L^p}=1}} \big|\langle T\widetilde{\overline{g}}, \widetilde{f}\rangle\big| \leq \sup_{\substack{g \in \mathscr{S} \\ \|g\|_{L^p}=1}} \|T\widetilde{\overline{g}}\|_{L^p}\,\|\widetilde{f}\|_{L^{p'}} \leq \|T\|_{L^p \to L^p}\,\|f\|_{L^{p'}},$$

using Hölder with exponents p and p' and $\|\widetilde{\overline{g}}\|_{L^p} = \|g\|_{L^p} = 1$. This holds for every f, so by definition of the operator norm $\|T\|_{L^{p'} \to L^{p'}} \leq \|T\|_{L^p \to L^p}$, which together with ($\dagger$) gives

$$\Longrightarrow \quad \|T\|_{L^p \to L^p} = \|T\|_{L^{p'} \to L^{p'}}.$$

The mechanism: an operator and its transpose have the same norm on dual spaces, and the transpose of $T : f \mapsto u * f$ is RTR with $R : f \mapsto \widetilde{f}$ the reflection, i.e. T conjugated by R; since R is an isometry on every L^q, RTR has the same norm as T.

Hence $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is bounded for $1 < p < \infty$. ∎

Note that we have shown

$$\|T\|_{L^p \to L^p} \;\le\; \begin{cases} \dfrac{B}{p-1} & \text{as } p \to 1^+ \\ C \cdot p & \text{as } p \to \infty \end{cases}$$

In the case of the Hilbert transform, $u = \frac{1}{\pi}\,\mathrm{p.v.}\frac{1}{x}$:

$$\|H\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \;=\; \begin{cases} \tan\dfrac{\pi}{2p}, & 1 < p \le 2 \\ \cot\Big(\dfrac{\pi}{2p}\Big), & 2 < p < \infty \end{cases} \qquad \textit{Pichorides 1972}$$

Note that we have also shown that T (and therefore H) is of weak type $(1,1)$. For H, this was proved by Kolmogorov A. in 1925.

Recall the partial sums of a Fourier series: $f : \mathbb{T} \to \mathbb{C}$,

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n)\, e^{2\pi i n x} \qquad \leftarrow \textit{linear}$$

$\|S_N f - f\|_{L^2} \to 0$ if $f \in L^2(\mathbb{T})$. What about pointwise a.e. convergence?

L. Carleson (1966): $S_N f(x) \to f(x)$ a.e. for $f \in L^2(\mathbb{T})$.

R. Hunt (1967): $S_N f(x) \to f(x)$ a.e. for $f \in L^p(\mathbb{T})$, $1 < p < \infty$.

Write

$$S^* f(x) = \sup_{k \in \mathbb{N}} \big|S_{2^k} f(x)\big| \qquad \leftarrow \textit{sublinear}$$

Littlewood–Paley square function

$$S_{\mathbb{T}} f(x) = \Big(|S_1 f(x)|^2 + \sum_{k=1}^{\infty} \big|S_{2^k} f(x) - S_{2^{k-1}} f(x)\big|^2\Big)^{1/2}$$

Littlewood–Paley showed in the 1930s:

$$(*) \qquad A_p \|f\|_{L^p(\mathbb{T})} \;\le\; \|S_{\mathbb{T}} f\|_{L^p(\mathbb{T})} \;\le\; B_p \|f\|_{L^p(\mathbb{T})}, \qquad p \in (1, \infty).$$

Let $\{a_k\}$ be a sequence in $\ell^2(\mathbb{N})$ such that

$$\widehat{f}(n) = \begin{cases} a_k & \text{if } n = 2^k \\ 0 & \text{otherwise} \end{cases}, \qquad \text{then} \qquad f = \sum_{n=0}^{\infty} \widehat{f}(n)\, e^{2\pi i n x} = \sum_{k=0}^{\infty} a_k\, e^{2\pi i\, 2^k x} \in L^2(\mathbb{T}),$$

by $\|f\|_{L^2} = \left(\sum_k |a_k|^2\right)^{1/2}$ (Parseval). By $(*)$

$$A_p\|f\|_{L^p(\mathbb{T})} \;\le\; \|S_{\mathbb{T}}f\|_{L^p(\mathbb{T})} \;\le\; B_p\|f\|_{L^p(\mathbb{T})}, \qquad S_{\mathbb{T}}f(x) = \|a\|_{\ell^2} \quad \forall\, x \in \mathbb{T}.$$

Why $S_{\mathbb{T}}f \equiv \|a\|_{\ell^2}$: the block $S_{2^k}f - S_{2^{k-1}}f$ collects the frequencies n with $2^{k-1} < |n| \le 2^k$, and among $\{2^j\}_{j\ge 0}$ exactly one lies in that range, namely $n = 2^k$. Hence

$$S_{2^k}f(x) - S_{2^{k-1}}f(x) = a_k\, e^{2\pi i\, 2^k x}, \qquad \left|S_{2^k}f(x) - S_{2^{k-1}}f(x)\right|^2 = |a_k|^2,$$

and likewise $S_1f(x) = a_0e^{2\pi ix}$, $|S_1f(x)|^2 = |a_0|^2$. The x-dependence sits in a unimodular factor and disappears under the modulus, so $S_{\mathbb{T}}f(x) = \left(\sum_{k\ge 0}|a_k|^2\right)^{1/2} = \|a\|_{\ell^2}$ for every x. This is the whole reason lacunary series are special: the gaps between frequencies are large enough that each dyadic block sees a single term.

That means for such an f all L^p norms are equivalent, we only pick up a constant based on p.

Hence

$$A_p\|f\|_{L^p(\mathbb{T})} \;\le\; \|a\|_{\ell^2} \;\le\; B_p\|f\|_{L^p(\mathbb{T})} \qquad \text{for } 1 < p < \infty.$$

(lacunary Fourier series).

Since $\|a\|_{\ell^2} = \|f\|_{L^2(\mathbb{T})}$ by Parseval, this reads

$$A_p\|f\|_{L^p(\mathbb{T})} \;\le\; \|f\|_{L^2(\mathbb{T})} \;\le\; B_p\|f\|_{L^p(\mathbb{T})} \qquad (1 < p < \infty)$$

for every lacunary f. In particular a lacunary $f \in L^2(\mathbb{T})$ lies in $L^p(\mathbb{T})$ for every *$p < \infty$, with $\|f\|_{L^p}$ controlled by $\|f\|_{L^2}$ alone, something a general L^2 function has no reason to satisfy. This is the classical fact that lacunary exponentials behave like independent random variables, in the same way Khintchine's inequality controls all L^p norms of Rademacher sums by the ℓ^2 norm of the coefficients.*