Harmonic Analysis

Lecture 7

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Exercise Classes: Thursdays 13.00–15.00

Littlewood–Paley Theory

$f \in L^1(\mathbb{T})$,

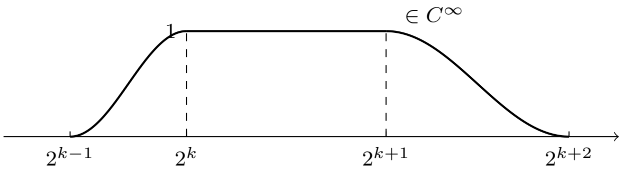
$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n)\, e^{2\pi i n x}, \qquad \Delta_k f(x) = S_{2^{k+1}} f(x) - S_{2^k} f(x).$$

(different S)

$$S_{\mathbb{T}} f(x) = \Big(|S_1 f(x)|^2 + \sum_{k=0}^{\infty} \big| S_{2^{k+1}} f(x) - S_{2^k} f(x) \big|^2 \Big)^{1/2} = \Big(|S_1 f(x)|^2 + \sum_{k=0}^{\infty} |\Delta_k f(x)|^2 \Big)^{1/2}.$$

We are after: for $p \in (1,\infty)$, $\exists\, A_p, B_p > 0$ such that

$$A_p \|f\|_{L^p(\mathbb{T})} \;\le\; \|S_{\mathbb{T}} f\|_{L^p(\mathbb{T})} \;\le\; B_p \|f\|_{L^p(\mathbb{T})}.$$



these will meet CZO and we know they are bounded.

Let $\eta \in C_c^\infty(\mathbb{R}^d)$ such that

- $\eta(\xi) = 1$ if $|\xi| \le 1$
- $\eta(\xi) = 0$ if $|\xi| \ge 2$
- $|\eta(\xi)| \le 1 \;\; \forall\, \xi \in \mathbb{R}^d$

$$\delta(\xi) = \eta(\xi) - \eta(2\xi) \qquad \forall\, \xi \in \mathbb{R}^d,$$

note that $\displaystyle\sum_{j=-\infty}^{\infty} \delta(2^{-j}\xi) = 1 \;\; \forall\, \xi \in \mathbb{R}^d \setminus \{0\}$. Look at

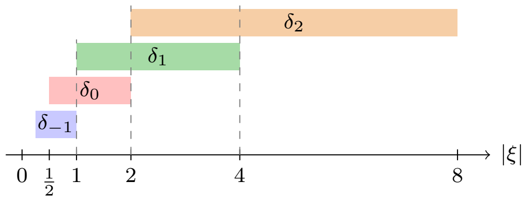
$$\sum_{j=-N}^{M} \delta(2^{-j}\xi) = \underbrace{\sum_{j=-N}^{M} \big(\eta(2^{-j}\xi) - \eta(2^{-j+1}\xi) \big)}_{\textit{telescoping}} = \underbrace{\eta(2^{-M}\xi)}_{\substack{\downarrow\, M\to\infty \\ 1}} - \underbrace{\eta(2^{N+1}\xi)}_{\substack{\downarrow\, N\to\infty \\ 0}}$$

$N, M \to \infty$, so $\to 1$.

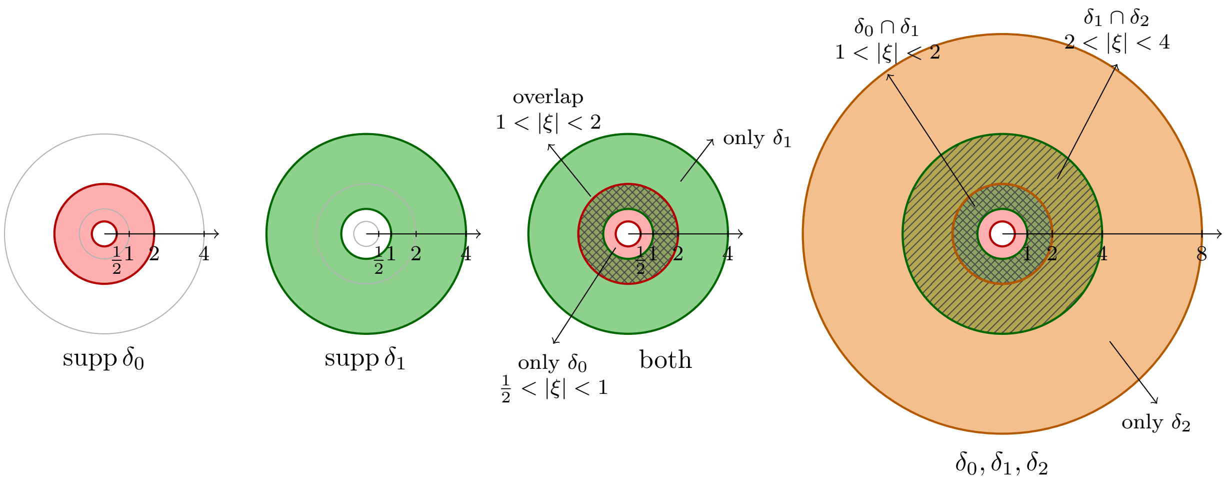
We call $\delta(2^{-j}\xi) =: \delta_j(\xi)$. Note

$$\operatorname{supp} \delta_j \subseteq \big\{ \xi \in \mathbb{R}^d : 2^{j-1} < |\xi| < 2^{j+1} \big\},$$

so live on dyadic shells of thickness 2^j.



supports of δ_j on the line: δ_j lives on $(2^{j-1}, 2^{j+1})$, so δ_j and δ_{j+1} overlap exactly on $(2^j, 2^{j+1})$, while δ_j and δ_{j+2} only touch at the point 2^{j+1}.



in two dimensions we get annuli. First: $\operatorname{supp}\delta_0 = \{\frac{1}{2} < |\xi| < 2\}$. *Second:* $\operatorname{supp}\delta_1 = \{1 < |\xi| < 4\}$. *Third: both together, the crosshatched ring $\{1 < |\xi| < 2\}$ is where they overlap. Fourth: $\delta_0, \delta_1, \delta_2$ together, with the two overlap rings hatched differently; δ_1 is entirely covered by its two neighbours, and δ_0, δ_2 do not overlap at all, they only touch along $|\xi| = 2$. Each δ_j overlaps its inner neighbour on the inner half of its ring and its outer neighbour on the outer half, so at no point are three of them nonzero, and δ_j, δ_{j+2} meet only along the circle $|\xi| = 2^{j+1}$. On $|\xi| = 2^j$ itself only δ_j is nonzero, and there $\delta_j = 1$, consistent with $\sum_j \delta_j = 1$.*

We now fix η, δ as above and define

$$\Delta_j f = \check{\delta}_j * f, \qquad f \in \mathscr{S}(\mathbb{R}^d).$$

Define Littlewood–Paley square function

$$Sf = \Big(\sum_{j=-\infty}^{\infty} |\Delta_j f|^2 \Big)^{1/2}$$

we want to prove:

<u>Theorem</u> For $p \in (1, \infty)$, there exist constants $A_{p,d,\eta}, B_{p,d,\eta} > 0$ such that

$$A_{p,d,\eta} \|f\|_{L^p(\mathbb{R}^d)} \;\le\; \underbrace{\|Sf\|_{L^p(\mathbb{R}^d)} \;\le\; B_{p,d,\eta} \|f\|_{L^p(\mathbb{R}^d)}}_{\textit{this part first}}.$$

<u>Technique</u>: prove the second part and use duality. (Note Sf is sublinear.)

First let us build some machinery.

Khintchine's inequalities

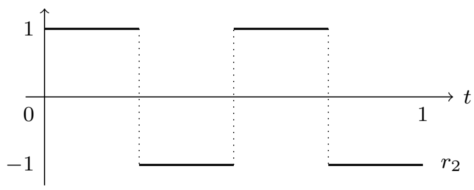
Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. We say that the independent random variables $r_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}_0$, are Rademacher, if

$$\mathbb{P}\big(\{\omega \in \Omega : r_n(\omega) = 1\}\big) = \mathbb{P}\big(\{\omega \in \Omega : r_n(\omega) = -1\}\big) = \tfrac{1}{2} \qquad \forall\, n \in \mathbb{N}_0.$$

Note independence is then

$$\int_\Omega r_n(\omega)\, r_m(\omega)\, d\mathbb{P}(\omega) = \int_\Omega r_n(\omega)\, d\mathbb{P}(\omega) \int_\Omega r_m(\omega)\, d\mathbb{P}(\omega) = 0 \qquad \text{for } n \neq m.$$

So looks like orthogonality!



e.g. for r_1 we can halve the interval, or enlarge the intervals

$$r_n(t) = \Big(\operatorname{sign}\big(\sin(2\pi t \cdot 2^n)\big)\Big)_{n \geq 0}.$$

Khintchine's inequalities For each $p \in (0, \infty)$, there exist $A_p, B_p > 0$ such that for any $N \in \mathbb{N}$, $a_1, \dots, a_N \in \mathbb{C}$,

$$A_p \Big\| \sum_{n=1}^N a_n r_n(\omega) \Big\|_{L^p(\Omega)} \;\leq\; \Big(\sum_{n=1}^N |a_n|^2 \Big)^{1/2} \;\leq\; B_p \Big\| \sum_{n=1}^N a_n r_n(\omega) \Big\|_{L^p(\Omega)}.$$

Linearization of the Littlewood–Paley square function

Choose Rademacher functions $\{r_j\}_{j \in \mathbb{Z}}$ on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. For $\omega \in \Omega$, $N \in \mathbb{N}_0$, write

$$m_{\omega,N}(\xi) = \sum_{j=-N}^N r_j(\omega)\, \delta_j(\xi), \qquad \xi \in \mathbb{R}^d$$

$\Rightarrow m_{\omega,N} \in \mathscr{S}(\mathbb{R}^d)$. Let $T_{\omega,N} f := \check{m}_{\omega,N} * f \;\; (f \in \mathscr{S}(\mathbb{R}^d))$. Hence

$$T_{\omega,N} f = \sum_{j=-N}^N r_j(\omega)\, \check{\delta}_j * f = \sum_{j=-N}^N r_j(\omega)\, \Delta_j f,$$

so that's the linearisation. Now we will show they are uniformly bounded. We shall prove that there exists $A_{d,\eta} > 0$ such that

(1) $\|m_{\omega,N}\|_{L^\infty(\mathbb{R}^d)} \leq A_{d,\eta}$

(2) $\displaystyle\sup_{y \in \mathbb{R}^d} \int_{|x| > 2|y|} \big| \check{m}_{\omega,N}(x-y) - \check{m}_{\omega,N}(x) \big|\, dx \leq A_{d,\eta}.$

Once we have (1), (2) we apply the Theorem on CZO of convolution type and find: for any $p \in (1,\infty)$ there exists $C_{p,d,\eta} > 0$ with

$$\|T_{\omega,N} f\|_{L^p(\mathbb{R}^d)} \;\le\; C_{p,d,\eta}\, \|f\|_{L^p(\mathbb{R}^d)} \tag{$*$}$$

! clearly doesn't depend on N, ω !

Once we have (1), (2) and thus $(*)$ by the Theorem from last time, we get

$$\begin{aligned}
C^p_{p,d,\eta} \int_{\mathbb{R}^d} |f(x)|^p \, dx \quad &\underset{\mathbb{P}(\Omega)=1}{\underset{\uparrow}{=}} \quad \int_\Omega \Big(C^p_{p,d,\eta} \int_{\mathbb{R}^d} |f(x)|^p \, dx \Big) d\mathbb{P}(\omega) \\
&\underset{\textit{apply } (*)}{\underset{\uparrow}{\ge}} \quad \int_\Omega \int_{\mathbb{R}^d} |T_{\omega,N} f(x)|^p \, dx \, d\mathbb{P}(\omega) \\
&\underset{\textit{Tonelli}}{\underset{\uparrow}{=}} \quad \int_{\mathbb{R}^d} \underbrace{\int_\Omega \Big| \sum_{j=-N}^{N} r_j(\omega)\, \Delta_j f(x) \Big|^p d\mathbb{P}(\omega)\; dx}_{\textit{use Khintchine}} \\
&\underset{\substack{\textit{Khintchine, right} \\ \textit{inequality, } a_j = \Delta_j f(x)}}{\underset{\uparrow}{\ge}} \quad B_p^{-p} \int_{\mathbb{R}^d} \Big(\sum_{j=-N}^{N} |\Delta_j f(x)|^2 \Big)^{p/2} dx \\
&\underset{N\to\infty}{\Longrightarrow} \quad \|Sf\|_{L^p} \;\le\; B_p C_{p,d,\eta}\, \|f\|_{L^p}.
\end{aligned}$$

This is exactly the 2nd half of the inequality.

The passage $N \to \infty$ in detail. For each N the chain gives

$$\int_{\mathbb{R}^d} g_N(x)\, dx \;\le\; B_p^p\, C^p_{p,d,\eta}\, \|f\|^p_{L^p}, \qquad g_N(x) := \Big(\sum_{j=-N}^{N} |\Delta_j f(x)|^2 \Big)^{p/2},$$

with a right-hand side that does not depend on N. This is the whole point of having $(*)$ uniform in N and ω. The functions g_N are nonnegative and increasing in N: passing from N to $N+1$ adds the nonnegative terms $|\Delta_{\pm(N+1)} f(x)|^2$ inside the sum, and $t \mapsto t^{p/2}$ is increasing on $[0,\infty)$. Hence $g_N(x) \to \big(\sum_{j\in\mathbb{Z}} |\Delta_j f(x)|^2\big)^{p/2} = (Sf(x))^p$ for every x, a priori with value in $[0,\infty]$. By monotone convergence,

$$\int_{\mathbb{R}^d} (Sf(x))^p \, dx = \lim_{N\to\infty} \int_{\mathbb{R}^d} g_N(x)\, dx \;\le\; B_p^p\, C^p_{p,d,\eta}\, \|f\|^p_{L^p}.$$

In particular the series $\sum_j |\Delta_j f(x)|^2$ converges for a.e. x, so Sf is finite a.e. and $Sf \in L^p(\mathbb{R}^d)$, which was not clear from the definition. Taking p-th roots gives $\|Sf\|_{L^p} \le B_p C_{p,d,\eta} \|f\|_{L^p}$.

We need to prove (1), (2).

For (1):

$$|m_{\omega,N}(\xi)| \quad \underset{\textit{triangle}}{\underset{\uparrow}{\le}} \quad \sum_{j=-N}^{N} \underbrace{|r_j(\omega)|}_{\le 1} \; \underbrace{|\delta_j(\xi)|}_{\le 2\|\eta\|_{L^\infty}} \;\le\; 4\, \|\eta\|_{L^\infty}$$

since δ_j: each shell overlaps only the neighbours. For fixed ξ, the j with $2^{j-1} < |\xi| < 2^{j+1}$ satisfy $|j - \log_2 |\xi|| < 1$, an open interval of length 2, so at most two terms are nonzero at any ξ.

For (2): Note that for $g \in \mathscr{S}(\mathbb{R}^d)$, $a, b \in \mathbb{N}_0^d$, we have that

$$(-2\pi i x)^b\, \partial^a \check{g}(x) = \int_{\mathbb{R}^d} \partial^b \big(P_a \cdot g\big)(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi, \qquad P_a = (i2\pi\xi)^a.$$

(∂_x^a under the integral sign brings down $(2\pi i\xi)^a = P_a$; each x_i is then produced by integrating by parts once in ξ_i, which costs a factor $-2\pi i$ and moves one derivative onto $P_a g$. Boundary terms vanish since $g \in \mathscr{S}$.) One can also show that

$$|x|^M \le C_{d,M} \sum_{|b|=M} |x^b|, \qquad |b| = b_1 + \cdots + b_d,$$

$$\rightsquigarrow \qquad (2\pi)^M |x^b| \left|\partial^a \check{\delta}_j(x)\right| \underset{\substack{\uparrow \\ \text{identity with } g=\delta_j,\ |b|=M, \\ \text{absolute values, } |e^{2\pi i x\cdot\xi}|=1,\ \text{supp}\,\delta_j}}{\le} \int_{2^{j-1}\le|\xi|\le 2^{j+1}} \left|\partial^b (P_a\delta_j)(\xi)\right| d\xi \qquad \forall\, b.$$

$$\underset{\substack{\text{sum over } |b|=M, \\ |x|^M \le C_{d,M}\sum |x^b|}}{\Longrightarrow} \quad \left|\partial^a \check{\delta}_j(x)\right| \;\le\; C'_{d,M}\, |x|^{-M} \sum_{|b|=M} \int_{2^{j-1}\le|\xi|\le 2^{j+1}} \left|\partial^b (P_a\delta_j)(\xi)\right| d\xi$$

$$\Longrightarrow \quad \left|\partial^a \check{m}_{\omega,N}(x)\right| \;\le\; B_{d,\eta,a}\, |x|^{-(|a|+d)} \qquad \forall\, x \in \mathbb{R}^d \setminus \{0\}$$

this gives (2).

We have thus shown: there exists $B_{p,\eta,d} > 0$ such that for $p \in (1,\infty)$

$$\|Sf\|_{L^p(\mathbb{R}^d)} \;\le\; B_{p,\eta,d}\, \|f\|_{L^p(\mathbb{R}^d)}.$$

<u>Lemma:</u> Let $f, g \in \mathscr{S}(\mathbb{R}^d)$, $k \in \mathbb{Z}$. Then

$$\int_{\mathbb{R}^d} \Delta_k f(x)\, \Delta_\ell g(x)\, dx = 0 \qquad \text{for } \ell \in \mathbb{Z} \text{ with } |k-\ell| > 1.$$

"so this is orthogonality except for neighbouring shells"

<u>Proof:</u> fix $k \in \mathbb{Z}$. Then, by the multiplication formula $\int_{\mathbb{R}^d} F(x)G(x)\,dx = \int_{\mathbb{R}^d} \widehat{F}(\xi)\,\widehat{G}(-\xi)\,d\xi$,

$$\int_{\mathbb{R}^d} \Delta_k f(x)\, \Delta_\ell g(x)\, dx = \int_{\mathbb{R}^d} \widehat{\Delta_k f}(\xi)\, \widehat{\Delta_\ell g}(-\xi)\, d\xi$$

$$\underset{\substack{\uparrow \\ \widehat{\Delta_k f} = \delta_k \widehat{f}}}{=} \int_{\mathbb{R}^d} \delta_k(\xi)\, \widehat{f}(\xi)\, \delta_\ell(-\xi)\, \widehat{g}(-\xi)\, d\xi = 0 \qquad \text{for } |\ell - k| > 1,$$

($\widehat{\Delta_k f} = \delta_k \widehat{f}$ because $\Delta_k f = \check{\delta}_k * f$ and the Fourier transform turns convolution into a product.) since $\operatorname{supp}\delta_j \subseteq \{\xi : 2^{j-1} < |\xi| < 2^{j+1}\}$, a set symmetric under $\xi \mapsto -\xi$, and the shells for k and ℓ are disjoint when $|k-\ell| > 1$. ∎

<u>Lemma</u> Let $f \in \mathscr{S}(\mathbb{R}^d)$. Then

$$\Big\| \sum_{k=-N}^{N} \Delta_k f - f \Big\|_{L^2} \xrightarrow[N\to\infty]{} 0.$$

In particular for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$,

$$\Big\langle \sum_{k=-N}^{N} \Delta_k f, \varphi \Big\rangle \xrightarrow[N\to\infty]{} \langle f, \varphi \rangle \qquad \textit{(strong} \Rightarrow \textit{weak convergence).}$$

Proof:

$$\Big\|\sum_{k=-N}^{N}\Delta_k f - f\Big\|_{L^2}^2 \underset{\substack{\uparrow\\ Plancherel}}{=} \int_{\mathbb{R}^d}\Big|\sum_{k=-N}^{N}\delta_k(\xi)\,\widehat{f}(\xi)-\widehat{f}(\xi)\Big|^2 d\xi$$

$$= \int_{\mathbb{R}^d}|\widehat{f}(\xi)|^2\underbrace{\Big(\Big|\sum_{k=-N}^{N}\delta_k(\xi)-1\Big|\Big)^2}_{=0 \text{ for } 2^{-N+1}<|\xi|<2^N} d\xi$$

$$\underset{\substack{\uparrow\\ telescoping}}{\leq} \big(1+2\|\eta\|_{L^\infty}\big)^2\int_{\{\xi:\,|\xi|\leq 2^{-N+1}\text{ or }|\xi|\geq 2^N\}}|\widehat{f}(\xi)|^2\,d\xi \xrightarrow[N\to\infty]{} 0$$

The $\leq$*:* $\sum_{|k|\leq N}\delta_k(\xi)=\eta(2^{-N}\xi)-\eta(2^{N+1}\xi)$ *by the telescoping on page 1, so* $|\sum_{|k|\leq N}\delta_k - 1|\leq 1+2\|\eta\|_{L^\infty}$.

The $\to 0$ *by dominated convergence. Write* $E_N:=\{\xi:|\xi|\leq 2^{-N+1}\}\cup\{\xi:|\xi|\geq 2^N\}$ *and* $h_N:=|\widehat{f}|^2\chi_{E_N}$*, so the last integral is* $\int_{\mathbb{R}^d}h_N(\xi)\,d\xi$*. Pointwise limit: fix* $\xi\neq 0$*; as soon as* $2^{-N+1}<|\xi|<2^N$*, i.e. for all* N *large enough (depending on* ξ*),* $\xi\notin E_N$ *and* $h_N(\xi)=0$*. Hence* $h_N\to 0$ *pointwise on* $\mathbb{R}^d\setminus\{0\}$*, which is a.e. Domination:* $0\leq h_N\leq|\widehat{f}|^2$ *for every* N*, and* $|\widehat{f}|^2\in L^1(\mathbb{R}^d)$ *because* $\widehat{f}\in\mathscr{S}(\mathbb{R}^d)\subseteq L^2(\mathbb{R}^d)$*. Dominated convergence then gives* $\int_{\mathbb{R}^d}h_N(\xi)\,d\xi\to\int_{\mathbb{R}^d}0\,d\xi=0$ *as* $N\to\infty$*. (Equivalently: the sets* E_N *decrease to* $\{0\}$*, a null set, and* $\int_E|\widehat{f}|^2$ *is a finite measure of* E*, so it is continuous from above.)* ■

The first part of the inequality will be addressed using dual spaces in the next lecture.