# Harmonic Analysis

Lecture 8

Teodoras Paura

25 September 2026

Exercise Classes: Thursdays 13.00–15.00

$$\Delta_j f = \check{\delta}_j * f, \qquad \delta_j(\xi) = \eta(2^{-j}\xi) - \eta(2^{-j+1}\xi) \qquad \textit{$\eta$ was the smooth cutoff.}$$

$$Sf = \Big(\sum_{j=-\infty}^{\infty} |\Delta_j f(x)|^2\Big)^{1/2} \qquad \textit{Littlewood–Paley square function}$$

Let $p \in (1,\infty)$, then there exist $A_p, B_p > 0$ such that

$$A_p \|f\|_{L^p} \ \le\ \|Sf\|_{L^p(\mathbb{R}^d)} \ \le\ B_p \|f\|_{L^p}.$$

We have shown the right inequality for $p \in (1,\infty)$. We used

$$m_{N,\omega}(\xi) = \sum_{n=-N}^{N} r_n(\omega)\,\delta_n(\xi) \qquad (\xi \in \mathbb{R}^d)$$

with apparatus for CZO's.

<u>Lemma.</u> Let $f \in \mathscr{S}(\mathbb{R}^d)$. Then

$$\Big\| f - \sum_{j=-N}^{N} \Delta_j f \Big\|_{L^2} \xrightarrow[N\to\infty]{} 0.$$

In particular for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$,

$$\Big\langle f - \sum_{j=-N}^{N} \Delta_j f, \varphi \Big\rangle \xrightarrow[N\to\infty]{} 0$$

$$\implies \qquad \sum_{j=-N}^{N} \Delta_j f \xrightarrow[N\to\infty]{} f \ \text{ in the space of tempered distributions } \mathscr{S}(\mathbb{R}^d)'.$$

Now note that for $f, g \in \mathscr{S}(\mathbb{R}^d)$,

$$\begin{aligned}
\Big|\int_{\mathbb{R}^d} f(x)\,\overline{g}(x)\,dx\Big| = |\langle f, \overline{g}\rangle| &\underset{\substack{\uparrow \\ \textit{Lemma, } L^2 \textit{ convergence}}}{=} \Big|\lim_{N\to\infty} \Big\langle \sum_{j=-N}^{N} \Delta_j f, \overline{g} \Big\rangle\Big| \\
&\underset{\substack{\uparrow \\ \textit{Lemma for } \overline{g}}}{=} \lim_{N\to\infty}\lim_{M\to\infty} \Big|\Big\langle \sum_{j=-N}^{N} \Delta_j f, \sum_{\ell=-M}^{M} \Delta_\ell \overline{g} \Big\rangle\Big| \\
&\underset{\substack{\uparrow \\ \textit{triangle}}}{\le} \lim_{N\to\infty}\lim_{M\to\infty} \sum_{j=-N}^{N} \sum_{\ell=-M}^{M} |\langle \Delta_j f, \Delta_\ell \overline{g}\rangle|
\end{aligned}$$

*we showed that $\langle \Delta_k f, \Delta_\ell g\rangle = 0$ for $|k-\ell|>1$ (Lecture 7, Lemma).*

$$\leq \sum_{j=-\infty}^{\infty} \sum_{\ell=-1}^{1} \big|\langle \Delta_j f, \Delta_{j+\ell}\overline{g}\rangle\big| \underset{|\langle u,v\rangle| \leq \int |u||v|}{\underset{\uparrow}{\leq}} \sum_{\ell=-1}^{1} \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^d} |\Delta_j f(x)|\, |\Delta_{j+\ell}\overline{g}(x)|\, dx$$

$$\underset{\textit{Beppo Levi}}{\underset{\uparrow}{=}} \sum_{\ell=-1}^{1} \int_{\mathbb{R}^d} \sum_{j=-\infty}^{\infty} |\Delta_j f(x)|\, |\Delta_{j+\ell}\overline{g}(x)|\, dx$$

$$\underset{\textit{C–S in } \ell^2(\mathbb{Z}) \textit{ at each fixed } x}{\underset{\uparrow}{\leq}} \sum_{\ell=-1}^{1} \int_{\mathbb{R}^d} \Big(\sum_{j=-\infty}^{\infty} |\Delta_j f|^2\Big)^{1/2} \Big(\sum_{j=-\infty}^{\infty} |\Delta_{j+\ell}\overline{g}|^2\Big)^{1/2} dx$$

$$\underset{\textit{remove } \ell\textit{: index shift } j+\ell\to j}{\underset{\uparrow}{=}} \sum_{\ell=-1}^{1} \int_{\mathbb{R}^d} \Big(\sum_j |\Delta_j f|^2\Big)^{1/2} \Big(\sum_j |\Delta_j \overline{g}|^2\Big)^{1/2} dx$$

$$\underset{(*)}{\leq} 3 \int_{\mathbb{R}^d} (Sf)(x)\,(S\overline{g})(x)\, dx \underset{\textit{Hölder}}{\underset{\uparrow}{\leq}} 3\, \|Sf\|_{L^p(\mathbb{R}^d)}\, \|S\overline{g}\|_{L^{p'}(\mathbb{R}^d)}.$$

Now $f \in \mathscr{S}(\mathbb{R}^d)$

$$\implies \quad \|Sf\|_{L^p} \geq \sup_{\substack{g\in\mathscr{S}(\mathbb{R}^d)\\ \|g\|_{L^{p'}}=1}} \frac{1}{B_{p'}} \langle Sf, S\overline{g}\rangle \underset{(*)\ \textit{used}}{\underset{\uparrow}{\geq}} \frac{1}{3B_{p'}} \sup_{\substack{g\in\mathscr{S}(\mathbb{R}^d)\\ \|g\|_{L^{p'}}=1}} \big|\langle f, \overline{g}\rangle\big|$$

$$\underset{\textit{duality, density of Schwartz}}{\underset{\uparrow}{=}} \frac{1}{3B_{p'}} \|f\|_{L^p},$$

*we already know $\|S\overline{g}\|_{L^{p'}} \leq B_{p'}\|\overline{g}\|_{L^{p'}} = B_{p'}\|g\|_{L^{p'}} = B_{p'}$ (upper bound of Lecture 7 applied to $\overline{g}\in\mathscr{S}$), so Hölder gives $\|Sf\|_{L^p} \geq \langle Sf, S\overline{g}\rangle / B_{p'}$.*

so we have shown

$$\|Sf\|_{L^p} \geq \frac{1}{3B_{p'}} \|f\|_{L^p}.$$

Choose $A_p = \dfrac{1}{3B_{p'}}$ and thus

$$A_p \|f\|_{L^p} \leq \|Sf\|_{L^p} \leq B_p \|f\|_{L^p}. \qquad \blacksquare$$

*Using duality to turn an upper bound into a lower one is an important trick to have. Worth keeping in mind.*

<u>Recall Main Theorem</u> Let $T$ be a translation-invariant CZO (or a CZO of convolution type, that is, there exists $u \in \mathscr{S}(\mathbb{R}^d)'$ with $Tf = u * f$ for $f\in\mathscr{S}(\mathbb{R}^d)$). Suppose further that

(I) $\widehat{u}$ agrees with an $L^\infty$ function

(II) $u$ agrees on $\mathbb{R}^d\setminus\{0\}$ with a function $K \in L^1_{\mathrm{loc}}(\mathbb{R}^d\setminus\{0\})$ such that

$$A = \sup_{y\in\mathbb{R}^d} \int_{|x|>2|y|} \big|K(x-y)-K(x)\big|\, dx < \infty.$$

Then $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ bounded for $p \in (1,\infty)$ **and**

$$\big|\{x \in \mathbb{R}^d : |Tf(x)| > a\}\big| \;\le\; C \cdot \frac{\|f\|_{L^1}}{a} \qquad \textit{($T$ is weak type $(1,1)$).}$$

*$\uparrow$ $Tf \in L^{1,\infty}(\mathbb{R}^d)$ but in general $Tf \notin L^1(\mathbb{R}^d)$.*

$$f = g + b, \qquad b = \sum_{Q\in\mathcal{Q}} b_Q, \qquad \int_Q b_Q = 0.$$

<u>Definition.</u> Let $Q$ be a cube in $\mathbb{R}^d$ and let $a_Q$ be a locally integrable function supported on $Q$ such that

$$\int_Q |a_Q(x)|^2\,dx \;\le\; \frac{1}{|Q|}, \qquad \int_Q a_Q(x)\,dx = 0,$$

then we call $a_Q$ an $(L^2)$ <u>atom</u> on $Q$.

Recalling from the proof of the Main Theorem in Lecture 6, $Q^*$ the enlargement of $Q$:

$$\int_{\mathbb{R}^d\setminus Q^*} |T(a_Q)(x)|\,dx \underset{\substack{\uparrow\\ \textit{Lecture 6, Hörmander}\\ \textit{and } \int_Q a_Q = 0}}{\le} C\int_Q |a_Q(x)|\,dx \underset{\substack{\uparrow\\ (\dagger)\ \textit{below}}}{\le} C$$

*Here $C = A = \sup_y \int_{|x|>2|y|} |K(x-y) - K(x)|\,dx$, the Hörmander constant of (II).*

$$\begin{aligned}
\int_{Q^*} |T(a_Q)(x)|\,dx &\underset{\substack{\uparrow\\ C\text{–}S}}{\le} |Q^*|^{1/2}\Big(\int_{Q^*} |T(a_Q)(x)|^2\,dx\Big)^{1/2} \\
&\underset{\substack{\uparrow\\ \textit{$L^2$ bound, Lecture 5 Fact 1:}\\ \|Tv\|_{L^2} \le \|\widehat{u}\|_{L^\infty}\|v\|_{L^2}}}{\le} \|\widehat{u}\|_{L^\infty}\cdot|Q^*|^{1/2}\Big(\int_Q |a_Q(x)|^2\,dx\Big)^{1/2} \\
&\underset{\substack{\uparrow\\ \textit{atom: } \int_Q |a_Q|^2 \le |Q|^{-1}}}{\le} |Q^*|^{1/2}\,|Q|^{-1/2}\cdot\|\widehat{u}\|_{L^\infty} \underset{\substack{\uparrow\\ |Q^*| = 2^d d^{d/2}|Q|}}{=} \|\widehat{u}\|_{L^\infty}\cdot 2^{d/2}d^{d/4}.
\end{aligned}$$

Note we used

$$(\dagger) \qquad \int_Q |a_Q(x)|\,dx \;\le\; |Q|^{1/2}\Big(\int_Q |a_Q(x)|^2\,dx\Big)^{1/2} \;\le\; 1.$$

<u>Definition.</u> Define $X$ as the class of $f \in L^1(\mathbb{R}^d)$ such that there exists a sequence $(\mu_k) \in \ell^1(\mathbb{N})$ and a sequence $(Q_k)$ of cubes such that

$$f = \sum_{k=1}^{\infty} \mu_k\, a_{Q_k}, \qquad \text{where } a_{Q_k} \text{ are \underline{atoms} on } Q_k \text{ for each } k.$$

*"=": partial sums converge in $L^1$ norm to the $f$.*

We write

$$\|f\|_X = \inf\Big\{\sum_{k=1}^{\infty} |\mu_k| \;:\; f = \sum_{k=1}^{\infty} \mu_k a_{Q_k}\Big\}.$$

One can show that $(X, \|\cdot\|_X)$ is a Banach space.

<u>Theorem</u> Let $T$ be a translation-invariant CZO as in the previous Main Theorem (Lecture 5). Then $Tf \in L^1$ for each $f \in X$, and $T : X \to L^1(\mathbb{R}^d)$ is bounded.

<u>Proof:</u> Let $f \in X$. Let $(\mu_k) \in \ell^1(\mathbb{N})$, $a_{Q_k}$ a sequence of atoms such that $f = \sum_{k=1}^{\infty} \mu_k a_{Q_k}$ (in the sense of $L^1$-convergence of the series). So we know that $Tf$ is well-defined, since $T$ is weak $(1,1)$ *(the $L^1$ extension of Lecture 6)*. From the atom estimate above (far part $\le C$ by (†), near part $\le \|\widehat{u}\|_{L^\infty} 2^{d/2} d^{d/4}$),

$$\|T(a_{Q_k})\|_{L^1(\mathbb{R}^d)} \ \le\ C + \|\widehat{u}\|_{L^\infty} 2^{d/2} d^{d/4} \qquad \forall\, k,$$

so

$$\sum_{k=1}^{\infty} \underbrace{\mu_k}_{\ell^1(\mathbb{N})} \quad \underbrace{T(a_{Q_k})}_{\text{bounded sequence in } L^1} \qquad \text{converges in } L^1(\mathbb{R}^d) \text{ to some } g \in L^1(\mathbb{R}^d) \qquad \textit{(}L^1 \textit{ is complete).}$$

We claim that $Tf = g$. In this case

$$\|Tf\|_{L^1} \ \le\ \sum_k |\mu_k| \big(\|\widehat{u}\|_{L^\infty} 2^{d/2} d^{d/4} + C\big) \qquad \Longrightarrow \qquad \|Tf\|_{L^1} \ \le\ \big(\|\widehat{u}\|_{L^\infty} 2^{d/2} d^{d/4} + C\big) \|f\|_X.$$

Note that for any $\varepsilon > 0$, writing $Ta_k := T(a_{Q_k})$ and omitting "$x \in \mathbb{R}^d$" in the sets,

$$\begin{aligned}
\big|\{|(Tf)(x) - g(x)| > \varepsilon\}\big| &= \Big|\Big\{\Big|Tf(x) - \sum_{k=1}^{\infty} \mu_k T a_k(x)\Big| > \varepsilon\Big\}\Big| \\
&\underset{\substack{\uparrow \\ \textit{split at } \varepsilon/2}}{\le} \Big|\Big\{\Big|Tf(x) - \sum_{k\le N} \mu_k T a_k(x)\Big| > \tfrac{\varepsilon}{2}\Big\}\Big| + \Big|\Big\{\Big|\sum_{k>N} \mu_k T a_k(x)\Big| > \tfrac{\varepsilon}{2}\Big\}\Big| \\
&\underset{\substack{\uparrow \\ \textit{weak } (1,1) \textit{ on the first,} \\ \textit{Chebyshev on the second}}}{\le} \frac{C'}{\varepsilon/2}\Bigg[\underbrace{\int_{\mathbb{R}^d} \Big|f(x) - \sum_{k\le N} \mu_k a_{Q_k}(x)\Big|\,dx}_{\to 0 \text{ as } N\to\infty} + \underbrace{\int_{\mathbb{R}^d} \Big|\sum_{k>N} \mu_k T a_k(x)\Big|\,dx}_{\to 0\text{: tails go to 0 because } L^1}\Bigg]
\end{aligned}$$

*For the first term: $T$ is linear, so $Tf - \sum_{k\le N} \mu_k T(a_{Q_k}) = T\big(f - \sum_{k\le N} \mu_k a_{Q_k}\big)$, and weak $(1,1)$ applies to that. For the second: $\int |\sum_{k>N} \mu_k T(a_{Q_k})| \le \sum_{k>N} |\mu_k|\, C \to 0$.*

Epsilon is fixed, only then $N \to \infty$ kills both terms, and then we can say this works for any $\varepsilon$.

$$\Longrightarrow \quad \big|\{|Tf(x) - g(x)| > \varepsilon\}\big| = 0 \ \text{ for each } \varepsilon > 0 \qquad \Longrightarrow \qquad Tf = g \ \text{ in } L^1(\mathbb{R}^d). \qquad \blacksquare$$