

Notation

λ denotes Lebesgue measure on $\mathbb{R}^d$ and $|E| = \lambda(E)$. $B(x, r)$ is the open ball of radius $r > 0$ centred at x , and χ_E is the indicator function of a set E . For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ the centred and uncentred Hardy–Littlewood maximal functions are

$$M_c f(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| \, dy, \quad Mf(x) = \sup_{\substack{B \ni x \\ B \text{ open ball}}} \frac{1}{|B|} \int_B |f(y)| \, dy,$$

both with values in $[0, \infty]$.

Used facts from Integration Theory

- Lebesgue measure is translation invariant and $|B(x, cr)| = c^d |B(x, r)|$ for $c > 0$.
- $C_c(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$.
- M is of weak type $(1, 1)$: there is C_d with $|\{x : Mg(x) > a\}| \leq \frac{C_d}{a} \|g\|_1$ for all $g \in L^1(\mathbb{R}^d)$, $a > 0$.
- Markov's inequality: for measurable $g \geq 0$ and $a > 0$, $|\{x : g(x) > a\}| \leq a^{-1} \int g$. In particular an integrable nonnegative function is finite a.e.
- A continuous function with compact support is uniformly continuous on $\mathbb{R}^d$.

Exercise (1). Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. Show that

$$M_c f(x) \leq Mf(x) \leq 2^d M_c f(x) \quad \text{for a.e. } x \in \mathbb{R}^d.$$

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Exercise (2). Let $f \in L^1(\mathbb{R}^d)$.

- a) Show that Mf is lower semicontinuous.
- b) Conclude that for any $a > 0$, $E_a = \{x \in \mathbb{R}^d : Mf(x) > a\}$ is open.

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Exercise (3). Let $(X, \mathcal{A}, \mu) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda)$ and let $1 \leq q < r \leq \infty$. Show that $L^q(X) + L^r(X) \subseteq L^1_{\text{loc}}(\mathbb{R}^d)$.

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Exercise (4). Let $f \in L^1(\mathbb{R}^d)$, $f \geq 0$. For $r > 0$ and $x \in \mathbb{R}^d$ let

$$f_r(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy.$$

- a) Show that $\|f - f_r\|_1 \xrightarrow{r \rightarrow 0} 0$.
- b) Conclude that there exists a strictly decreasing sequence (r_n) , $r_n \xrightarrow{n \rightarrow \infty} 0$, such that $f_{r_n}(x) \rightarrow f(x)$ a.e. on $\mathbb{R}^d$.

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Exercise (5). Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and define f_r as in Question 4. For $x \in \mathbb{R}^d$ we define the oscillation of f at x ,

$$\text{osc}(f)(x) = \left| \limsup_{r \rightarrow 0} f_r(x) - \liminf_{r \rightarrow 0} f_r(x) \right|.$$

- a) Prove that $\text{osc}(f) \leq 2Mf$ a.e. on $\mathbb{R}^d$.
- b) Show that for any continuous function h on $\mathbb{R}^d$, $\text{osc}(h)$ is 0 on $\mathbb{R}^d$.
- c) Using the sublinearity of osc , show that for any continuous function h on $\mathbb{R}^d$ and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, $\text{osc}(f) \leq 2M(f - h)$.
- d) Use the density of $C_c(\mathbb{R}^d)$ in $L^1(\mathbb{R}^d)$ and the fact that M is weak type $(1, 1)$ to conclude that even for $f \in L^1(\mathbb{R}^d)$, $\text{osc}(f) = 0$ a.e. on $\mathbb{R}^d$.
- e) Conclude the Lebesgue Differentiation Theorem: for $f \in L^1(\mathbb{R}^d)$, $f_r \xrightarrow{r \rightarrow 0} f$ a.e.

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