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Harmonic Analysis, Autumn 2026 MATP32

Exercises 2 — Solutions

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Notation.

- λ is Lebesgue measure on $\mathbb{R}^d$, $|E| = \lambda(E)$, $B(x,r)$ the open ball, χ_E the indicator of E, $C_c(\mathbb{R}^d)$ the continuous compactly supported functions, $\mathscr{S}(\mathbb{R}^d)$ the Schwartz class.
- For $h : \mathbb{R}^d \to \mathbb{C}$ and $t > 0$: $h_t(x) := t^{-d}h(x/t)$. Then $\|h_t\|_{L^1} = \|h\|_{L^1}$ and $\{y : y/t \in B(0,\rho)\} = B(0,t\rho)$.
- Maximal functions (Lecture 1), for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$:

$$M_c f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy, \qquad Mf(x) = \sup_{\substack{B \ni x \\ B \text{ open ball}}} \frac{1}{|B|} \int_B |f(y)|\,dy.$$

  One has $M_c f \le Mf$.
- For measurable $F, H \ge 0$ the convolution $(F * H)(x) = \int_{\mathbb{R}^d} F(x-y)H(y)\,dy \in [0,\infty]$ is defined for every x.
- $|B(x,t\rho)| = t^d\,|B(0,\rho)|$ for $t, \rho > 0$.

Used facts from Integration Theory.

- *Tonelli.* If $g : X \times Y \to [0,\infty]$ is $\mathcal{A}\times\mathcal{B}$-measurable, then $y \mapsto \int_X g(x,y)\,d\mu(x)$ is $\mathcal{B}$-measurable, $x \mapsto \int_Y g(x,y)\,d\nu(y)$ is $\mathcal{A}$-measurable, and

$$\int_Y \int_X g\,d\mu\,d\nu = \int_X \int_Y g\,d\nu\,d\mu.$$

- *Fubini.* If $g : X \times Y \to \mathbb{C}$ is $\mathcal{A}\times\mathcal{B}$-measurable and $g(\cdot,y)$ is μ-integrable for ν-a.e. y, then the set of such y is in $\mathcal{B}$ and $y \mapsto \int_X g(x,y)\,d\mu(x)$ is $\mathcal{B}$-measurable on it.
- *Hölder.* For measurable $u, v : Y \to [0,\infty]$, $1 < p < \infty$ and q the conjugate exponent, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_Y uv\,d\nu \le \Big(\int_Y u^p\,d\nu\Big)^{1/p} \Big(\int_Y v^q\,d\nu\Big)^{1/q},$$

  with no integrability assumed (both sides may be ∞).
- *Monotone convergence.* If $0 \le g_1 \le g_2 \le \cdots$ are measurable and $g_n \to g$ pointwise as $n \to \infty$, then $\int_Y g_n\,d\nu \to \int_Y g\,d\nu$ as $n \to \infty$.
- *Dominated convergence.* If $g_n \to g$ pointwise a.e. as $n \to \infty$ and $|g_n| \le G$ a.e. for every n with $\int G < \infty$, then $\int g_n \to \int g$ as $n \to \infty$.
- *Dirichlet integral.* $S(t) := \int_0^t \frac{\sin s}{s}\,ds$ satisfies $0 \le S(t) \le S(\pi) < 2$ for $t \ge 0$ and $S(t) \to \frac{\pi}{2}$ as $t \to \infty$.

Used facts from the lectures.

- (L1) $M_c f \le Mf$ (Lecture 1) and M is weak type $(1,1)$: $|\{Mv > s\}| \le \frac{3^d}{s}\,\|v\|_{L^1}$ (Lecture 1, Theorem 1).

- (L2) M is strong type (p,p) for $1 < p < \infty$: $\|Mv\|_{L^p} \le C_{p,d}\,\|v\|_{L^p}$ (Lecture 2, Corollary). With Markov this gives, for every $1 \le p < \infty$ and $s > 0$,

$$|\{Mv > s\}| \le \frac{C_{p,d}^p}{s^p}\,\|v\|_{L^p}^p \qquad (C_{1,d} := 3^d).$$

- (L3) If $k \in L^1(\mathbb{R}^d)$ with $\int k = 1$, then $(k_t)_{t>0}$ is an approximate identity (Lecture 3, Example), and $k_t * h \to h$ uniformly as $t \to 0^+$ for $h \in C_0(\mathbb{R}^d)$ (Lecture 3, Theorem (ii)).
- (L4) Fourier transform $\widehat{\phi}(\xi) = \int_{\mathbb{R}} \phi(x) e^{-2\pi i x\xi}\,dx$, and $\langle \widehat{u}, \phi\rangle := \langle u, \widehat{\phi}\rangle$ for $u \in \mathscr{S}'$ (Lecture 4). Seminorms $p_{a,b}(\phi) = \sup_x |x^a \phi^{(b)}(x)|$, and $u : \mathscr{S} \to \mathbb{C}$ linear is bounded iff $|u(\phi)| \le C \sum_{a\le k,\, b\le \ell} p_{a,b}(\phi)$ (Lecture 3).

Exercise (1)**.** Prove the Minkowski integral inequality: let $1 \le p < \infty$ and let $f : X \times Y \to \mathbb{C}$ be $\mathcal{A}\times\mathcal{B}$-measurable with $f(\cdot, y)$ μ-integrable for ν-almost every $y \in Y$. Then

$$\Big(\int_Y \Big| \int_X f(x,y)\,d\mu(x) \Big|^p d\nu(y) \Big)^{1/p} \le \int_X \Big(\int_Y |f(x,y)|^p\,d\nu(y) \Big)^{1/p} d\mu(x).$$

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Exercise (2)**.** Let $g : [0,\infty) \to [0,\infty)$ such that g is decreasing and has only finitely many discontinuities.

a) Define $k : \mathbb{R}^d \to [0,\infty)$, $k(x) = g(|x|)$ for $x \in \mathbb{R}^d$. Suppose that $k \in L^1(\mathbb{R}^d)$. Show that for every $f \in L^1_{\text{loc}}(\mathbb{R}^d)$,

$$\sup_{t>0} \big(|f| * k_t\big)(x) \le \|k\|_{L^1}\ M_c f(x) \qquad \text{for a.e. } x \in \mathbb{R}^d.$$

b) Now suppose that $k \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} k(x)\,dx = 1$ and $|k(x)| \le g(|x|)$ for all $x \in \mathbb{R}^d$. Show that for any $p \in [1,\infty)$ and any $f \in L^p(\mathbb{R}^d)$,

$$\lim_{t\to 0^+} (k_t * f)(x) = f(x) \qquad \text{for a.e. } x \in \mathbb{R}^d.$$

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Exercise (3)**.** Let $p \in [1, \infty)$, $f \in L^p(\mathbb{R}^d)$. We define

$$u : \mathbb{R}^d \times (0, \infty) \to \mathbb{C}, \qquad u(x, t) = P_t * f(x),$$

where P is the Poisson kernel given by

$$P(x) = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \frac{1}{(1+|x|^2)^{(d+1)/2}} \qquad (x \in \mathbb{R}^d),$$

and $P_t(x) = t^{-d} P(x/t)$. Show that u solves the Dirichlet problem on the half-space $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times (0, \infty)$ with boundary values f in the following sense:

a) Show that u is harmonic on $\mathbb{R}^{d+1}_+$, i.e. $\partial_t^2 u(x,t) + \sum_{j=1}^d \partial_j^2 u(x,t) = 0$ for $(x,t) \in \mathbb{R}^{d+1}_+$.

b) Show that $\lim_{t \to 0^+} u(x,t) = f(x)$ for a.e. $x \in \mathbb{R}^d$.

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Exercise (4). a) Show that the functional p.v.$(1/x)$ given by

$$\left\langle \text{p.v.}\frac{1}{x}, \phi \right\rangle = \lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} \frac{\phi(x)}{x}\, dx \qquad (\phi \in \mathscr{S}(\mathbb{R}))$$

is well-defined (i.e. the limit above exists), and a bounded linear functional on $\mathscr{S}(\mathbb{R})$, hence a tempered distribution.

b) Show that $\widehat{\text{p.v.}(1/x)} = -i\pi \operatorname{sgn}$, i.e.

$$\left\langle \widehat{\text{p.v.}\frac{1}{x}}, \phi \right\rangle = \langle -i\pi \operatorname{sgn}, \phi \rangle \qquad (\phi \in \mathscr{S}(\mathbb{R})), \qquad \operatorname{sgn}(\xi) = \begin{cases} 1 & \xi > 0 \\ 0 & \xi = 0 \\ -1 & \xi < 0. \end{cases}$$

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