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Exercises 2 — Solutions

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Notation.

- λ is Lebesgue measure on $\mathbb{R}^d$, $|E| = \lambda(E)$, $B(x,r)$ the open ball, χ_E the indicator of E, $C_c(\mathbb{R}^d)$ the continuous compactly supported functions, $\mathscr{S}(\mathbb{R}^d)$ the Schwartz class.
- For $h : \mathbb{R}^d \to \mathbb{C}$ and $t > 0$: $h_t(x) := t^{-d}h(x/t)$. Then $\|h_t\|_{L^1} = \|h\|_{L^1}$ and $\{y : y/t \in B(0,\rho)\} = B(0,t\rho)$.
- Maximal functions (Lecture 1), for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$:

$$M_c f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy, \qquad Mf(x) = \sup_{\substack{B \ni x \\ B \text{ open ball}}} \frac{1}{|B|} \int_B |f(y)|\, dy.$$

  One has $M_c f \le Mf$.
- For measurable $F, H \ge 0$ the convolution $(F * H)(x) = \int_{\mathbb{R}^d} F(x-y)H(y)\, dy \in [0,\infty]$ is defined for every x.
- $|B(x,t\rho)| = t^d\, |B(0,\rho)|$ for $t, \rho > 0$.

Used facts from Integration Theory.

- *Tonelli.* If $g : X \times Y \to [0,\infty]$ is $\mathcal{A}\times\mathcal{B}$-measurable, then $y \mapsto \int_X g(x,y)\, d\mu(x)$ is $\mathcal{B}$-measurable, $x \mapsto \int_Y g(x,y)\, d\nu(y)$ is $\mathcal{A}$-measurable, and

$$\int_Y \int_X g\, d\mu\, d\nu = \int_X \int_Y g\, d\nu\, d\mu.$$

- *Fubini.* If $g : X \times Y \to \mathbb{C}$ is $\mathcal{A} \times \mathcal{B}$-measurable and $g(\cdot, y)$ is μ-integrable for ν-a.e. y, then the set of such y is in $\mathcal{B}$ and $y \mapsto \int_X g(x,y)\, d\mu(x)$ is $\mathcal{B}$-measurable on it.

- *Hölder.* For measurable $u, v : Y \to [0,\infty]$, $1 < p < \infty$ and q the conjugate exponent, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_Y uv\, d\nu \le \Big(\int_Y u^p\, d\nu \Big)^{1/p} \Big(\int_Y v^q\, d\nu \Big)^{1/q},$$

  with no integrability assumed (both sides may be ∞).

- *Monotone convergence.* If $0 \le g_1 \le g_2 \le \cdots$ are measurable and $g_n \to g$ pointwise as $n \to \infty$, then $\int_Y g_n\, d\nu \to \int_Y g\, d\nu$ as $n \to \infty$.

- *Dominated convergence.* If $g_n \to g$ pointwise a.e. as $n \to \infty$ and $|g_n| \le G$ a.e. for every n with $\int G < \infty$, then $\int g_n \to \int g$ as $n \to \infty$.

- *Dirichlet integral.* $S(t) := \int_0^t \frac{\sin s}{s}\, ds$ satisfies $0 \le S(t) \le S(\pi) < 2$ for $t \ge 0$ and $S(t) \to \frac{\pi}{2}$ as $t \to \infty$.

Used facts from the lectures.

- (L1) $M_c f \le Mf$ (Lecture 1) and M is weak type $(1,1)$: $|\{Mv > s\}| \le \frac{3^d}{s} \|v\|_{L^1}$ (Lecture 1, Theorem 1).

- (L2) M is strong type (p,p) for $1 < p < \infty$: $\|Mv\|_{L^p} \le C_{p,d}\,\|v\|_{L^p}$ (Lecture 2, Corollary). With Markov this gives, for every $1 \le p < \infty$ and $s > 0$,

$$|\{Mv > s\}| \le \frac{C_{p,d}^p}{s^p}\,\|v\|_{L^p}^p \qquad (C_{1,d} := 3^d).$$

- (L3) If $k \in L^1(\mathbb{R}^d)$ with $\int k = 1$, then $(k_t)_{t>0}$ is an approximate identity (Lecture 3, Example), and $k_t * h \to h$ uniformly as $t \to 0^+$ for $h \in C_0(\mathbb{R}^d)$ (Lecture 3, Theorem (ii)).
- (L4) Fourier transform $\widehat{\phi}(\xi) = \int_{\mathbb{R}} \phi(x) e^{-2\pi i x\xi}\,dx$, and $\langle \widehat{u}, \phi\rangle := \langle u, \widehat{\phi}\rangle$ for $u \in \mathscr{S}'$ (Lecture 4). Seminorms $p_{a,b}(\phi) = \sup_x |x^a \phi^{(b)}(x)|$, and $u : \mathscr{S} \to \mathbb{C}$ linear is bounded iff $|u(\phi)| \le C \sum_{a\le k,\, b\le \ell} p_{a,b}(\phi)$ (Lecture 3).

Exercise (1)**.** Prove the Minkowski integral inequality: let $1 \le p < \infty$ and let $f : X \times Y \to \mathbb{C}$ be $\mathcal{A} \times \mathcal{B}$-measurable with $f(\cdot, y)$ μ-integrable for ν-almost every $y \in Y$. Then

$$\Big(\int_Y \Big|\int_X f(x,y)\,d\mu(x)\Big|^p d\nu(y)\Big)^{1/p} \le \int_X \Big(\int_Y |f(x,y)|^p\,d\nu(y)\Big)^{1/p} d\mu(x).$$

Solution. By assumption $f(\cdot,y)$ is μ-integrable for ν-almost every $y \in Y$, so

$$F(y) := \Big|\int_X f(x,y)\,d\mu(x)\Big|$$

is defined for ν-almost every $y \in Y$ and is $\mathcal{B}$-measurable by Fubini's theorem. Since the absolute value of an integral is at most the integral of the absolute value,

$$F(y) \le \int_X |f(x,y)|\,d\mu(x) \qquad \text{for } \nu\text{-almost every } y \in Y. \tag{1}$$

The claim is

$$\Big(\int_Y F(y)^p\,d\nu(y)\Big)^{1/p} \le \int_X \Big(\int_Y |f(x,y)|^p\,d\nu(y)\Big)^{1/p} d\mu(x). \tag{2}$$

Case $p = 1$**.** Integrating (1) over Y and applying Tonelli's theorem to the nonnegative measurable function $|f|$,

$$\int_Y F(y)\,d\nu(y) \le \int_Y \int_X |f(x,y)|\,d\mu(x)\,d\nu(y) = \int_X \int_Y |f(x,y)|\,d\nu(y)\,d\mu(x),$$

which is (2) for $p = 1$.

Case $1 < p < \infty$**.** Let q be the conjugate exponent of p, so that $(p-1)q = p$ and $1 - \frac{1}{q} = \frac{1}{p}$. Since ν is σ-finite there are sets $E_1 \subseteq E_2 \subseteq \cdots$ in $\mathcal{B}$ with $\nu(E_n) < \infty$ for every n and $\bigcup_{n\ge 1} E_n = Y$. For $n \ge 1$ define

$$F_n(y) := \min\{F(y), n\}\,\chi_{E_n}(y) \qquad (y \in Y).$$

Each F_n is $\mathcal{B}$-measurable, $0 \le F_n \le F$, and $F_n(y) \to F(y)$ monotonically as $n \to \infty$ for ν-almost every $y \in Y$. Moreover

$$\int_Y F_n(y)^p\,d\nu(y) \le n^p\,\nu(E_n) < \infty. \tag{3}$$

Fix $n \ge 1$. We write $F_n^p = F_n^{p-1} F_n$ and estimate the second factor. By $F_n \le F$ and (1),

$$\int_Y F_n(y)^p\,d\nu(y) = \int_Y F_n(y)^{p-1} F_n(y)\,d\nu(y) \le \int_Y F_n(y)^{p-1} \int_X |f(x,y)|\,d\mu(x)\,d\nu(y).$$

The function $(x,y) \mapsto F_n(y)^{p-1}|f(x,y)|$ is nonnegative and $\mathcal{A}\times\mathcal{B}$-measurable, so by Tonelli's theorem

$$\int_Y F_n(y)^{p-1} \int_X |f(x,y)|\,d\mu(x)\,d\nu(y) = \int_X \int_Y F_n(y)^{p-1}|f(x,y)|\,d\nu(y)\,d\mu(x).$$

For fixed $x \in X$, Hölder's inequality in the variable y with exponent q on F_n^{p-1} and exponent p on $|f(x,\cdot)|$ gives, using $(p-1)q = p$,

$$\int_Y F_n(y)^{p-1}|f(x,y)|\,d\nu(y) \le \Big(\int_Y F_n(y)^p\,d\nu(y)\Big)^{1/q}\Big(\int_Y |f(x,y)|^p\,d\nu(y)\Big)^{1/p}.$$

The first factor on the right does not depend on x. Integrating over X and combining the three displays,

$$\int_Y F_n(y)^p\,d\nu(y) \le \Big(\int_Y F_n(y)^p\,d\nu(y)\Big)^{1/q}\int_X\Big(\int_Y |f(x,y)|^p\,d\nu(y)\Big)^{1/p}d\mu(x). \tag{4}$$

If $\int_Y F_n(y)^p\,d\nu(y) = 0$, then the left-hand side of (5) below is 0 and there is nothing to prove. Otherwise $0 < \int_Y F_n(y)^p\,d\nu(y) < \infty$ by (3), so we may divide (4) by $\big(\int_Y F_n(y)^p\,d\nu(y)\big)^{1/q}$, and since $1 - \frac{1}{q} = \frac{1}{p}$ we obtain

$$\Big(\int_Y F_n(y)^p\,d\nu(y)\Big)^{1/p} \le \int_X\Big(\int_Y |f(x,y)|^p\,d\nu(y)\Big)^{1/p}d\mu(x) \qquad \text{for every } n \ge 1. \tag{5}$$

Finally, F_n^p is nonnegative, measurable and increasing in n, with $F_n(y)^p \to F(y)^p$ as $n \to \infty$ for ν-almost every $y \in Y$. By the monotone convergence theorem,

$$\int_Y F_n(y)^p\,d\nu(y) \to \int_Y F(y)^p\,d\nu(y) \qquad \text{as } n \to \infty,$$

and since the right-hand side of (5) does not depend on n, letting $n \to \infty$ in (5) yields (2).

Exercise (2)**.** Let $g : [0,\infty) \to [0,\infty)$ such that g is decreasing and has only finitely many discontinuities.

a) Define $k : \mathbb{R}^d \to [0,\infty)$, $k(x) = g(|x|)$ for $x \in \mathbb{R}^d$. Suppose that $k \in L^1(\mathbb{R}^d)$. Show that for every $f \in L^1_{\text{loc}}(\mathbb{R}^d)$,

$$\sup_{t>0}\big(|f| * k_t\big)(x) \le \|k\|_{L^1}\ M_c f(x) \qquad \text{for a.e. } x \in \mathbb{R}^d.$$

b) Now suppose that $k \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} k(x)\,dx = 1$ and $|k(x)| \le g(|x|)$ for all $x \in \mathbb{R}^d$. Show that for any $p \in [1,\infty)$ and any $f \in L^p(\mathbb{R}^d)$,

$$\lim_{t\to 0^+}(k_t * f)(x) = f(x) \qquad \text{for a.e. } x \in \mathbb{R}^d.$$

Solution. **a)** *Step 1: k is an increasing limit of sums of ball indicators.* For $n \in \mathbb{N}$ and $1 \le m \le n2^n$ put

$$E_{m,n} := \{r \ge 0 : g(r) > m2^{-n}\}, \qquad \rho_{m,n} := \sup E_{m,n} \quad (\rho_{m,n} := 0 \text{ if } E_{m,n} = \emptyset).$$

Since g is decreasing, $E_{m,n} = [0,\rho_{m,n})$ or $[0,\rho_{m,n}]$, and $\rho_{m,n} < \infty$ because otherwise $k \ge m2^{-n} > 0$ on $\mathbb{R}^d$, contradicting $k \in L^1$. Let

$$g_n := 2^{-n}\sum_{m=1}^{n2^n}\chi_{E_{m,n}}, \qquad k_n(x) := g_n(|x|) = 2^{-n}\sum_{m=1}^{n2^n}\chi_{B(0,\rho_{m,n})}(x) \ \text{ for a.e. } x.$$

Then $0 \le g_n \le g_{n+1} \le g$ and $g_n \to g$ pointwise as $n \to \infty$, hence $0 \le k_n \le k$, $k_n \to k$ pointwise, and

$$\|k_n\|_{L^1} = 2^{-n} \sum_{m=1}^{n2^n} |B(0, \rho_{m,n})| \le \|k\|_{L^1} \,.$$

Step 2: the estimate for one ball. For $\rho, t > 0$ and every x, with $|B(x, t\rho)| = t^d \, |B(0, \rho)|$,

$$\begin{aligned}\int_{\mathbb{R}^d} |f(x-y)| \, t^{-d} \chi_{B(0,\rho)}(y/t) \, dy &= t^{-d} \int_{B(x,t\rho)} |f(z)| \, dz \\ &= |B(0,\rho)| \, \frac{1}{|B(x,t\rho)|} \int_{B(x,t\rho)} |f(z)| \, dz \le |B(0,\rho)| \; M_c f(x).\end{aligned}$$

Step 3: sum and pass to the limit. Summing Step 2 over the balls in k_n,

$$\big(|f| * (k_n)_t\big)(x) \le 2^{-n} \sum_{m=1}^{n2^n} |B(0, \rho_{m,n})| \; M_c f(x) = \|k_n\|_{L^1} \, M_c f(x) \le \|k\|_{L^1} \, M_c f(x).$$

For fixed t, x, the integrands $|f(x-y)| (k_n)_t(y)$ increase to $|f(x-y)| k_t(y)$ as $n \to \infty$, so by monotone convergence

$$\big(|f| * k_t\big)(x) = \lim_{n \to \infty} \big(|f| * (k_n)_t\big)(x) \le \|k\|_{L^1} \, M_c f(x) \qquad \text{for every } x \in \mathbb{R}^d \text{ and } t > 0.$$

Take $\sup_{t>0}$. (The bound holds for every x, in particular a.e.)

b) Put $G(x) := g(|x|)$, so $G \in L^1(\mathbb{R}^d)$ as in a), and $|k_t| \le G_t$ for all $t > 0$. Fix $p \in [1, \infty)$, $f \in L^p(\mathbb{R}^d)$.

Step 1: $k_t * f$ is defined a.e. and dominated. By a) applied to G, for every x and $t > 0$,

$$\int_{\mathbb{R}^d} |k_t(y)| \, |f(x-y)| \, dy \;\le\; \big(|f| * G_t\big)(x) \;\le\; \|G\|_{L^1} \; M_c f(x) \;\le\; \|G\|_{L^1} \; M f(x). \qquad (6)$$

By (L1), (L2), $Mf < \infty$ a.e., so $(k_t * f)(x) = \int k_t(y) f(x-y) \, dy$ converges absolutely for a.e. x, and $|(k_t * f)(x)|$ is bounded by (6). (By Exercise 1, moreover $\|k_t * f\|_{L^p} \le \|k\|_{L^1} \, \|f\|_{L^p}$.)

Step 2: continuous compactly supported h. By (L3), for $h \in C_c(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)$, $(k_t * h)(x) \to h(x)$ as $t \to 0^+$ for every x.

Step 3: oscillation. For $F \in L^p(\mathbb{R}^d)$ set $\Theta F(x) := \limsup_{t \to 0^+} \big|(k_t * F)(x) - F(x)\big| \in [0, \infty]$. Step 2 gives $\Theta h = 0$ for $h \in C_c(\mathbb{R}^d)$. For such h, by linearity of $*$, the triangle inequality, and (6) applied to $f - h$,

$$\Theta f \;\le\; \Theta(f-h) + \Theta h \;=\; \Theta(f-h) \;\le\; \sup_{t>0} |k_t * (f-h)| + |f-h| \;\le\; \|G\|_{L^1} \; M(f-h) + |f-h| \qquad \text{a.e.} \qquad (7)$$

Step 4: the bad set is null. Fix $a > 0$, $h \in C_c(\mathbb{R}^d)$. By (7), Markov, and (L2),

$$\{\Theta f > a\} \subseteq A_h := \Big\{ M(f-h) > \tfrac{a}{2\|G\|_{L^1}} \Big\} \cup \Big\{ |f-h| > \tfrac{a}{2} \Big\},$$

$$|A_h| \le \Big[\Big(\frac{2 \, \|G\|_{L^1} \, C_{p,d}}{a} \Big)^p + \Big(\frac{2}{a} \Big)^p \Big] \, \|f - h\|_{L^p}^p \,.$$

Choose $h_n \in C_c(\mathbb{R}^d)$ with $\|f - h_n\|_{L^p} \to 0$ as $n \to \infty$ (density of C_c in L^p). Then $\{\Theta f > a\} \subseteq \bigcap_n A_{h_n}$, a set of measure $\le \inf_n |A_{h_n}| = 0$. Hence $\{\Theta f > 0\} = \bigcup_{j \ge 1} \{\Theta f > 1/j\}$ is contained in a null set, i.e. $\Theta f = 0$ a.e., which is the claim.

Exercise (3)**.** Let $p \in [1,\infty)$, $f \in L^p(\mathbb{R}^d)$. We define

$$u : \mathbb{R}^d \times (0,\infty) \to \mathbb{C}, \qquad u(x,t) = P_t * f(x),$$

where P is the Poisson kernel given by

$$P(x) = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \frac{1}{(1+|x|^2)^{(d+1)/2}} \qquad (x \in \mathbb{R}^d),$$

and $P_t(x) = t^{-d}P(x/t)$. Show that u solves the Dirichlet problem on the half-space $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times (0,\infty)$ with boundary values f in the following sense:

a) Show that u is harmonic on $\mathbb{R}^{d+1}_+$, i.e. $\partial_t^2 u(x,t) + \sum_{j=1}^d \partial_j^2 u(x,t) = 0$ for $(x,t) \in \mathbb{R}^{d+1}_+$.

b) Show that $\lim_{t\to 0^+} u(x,t) = f(x)$ for a.e. $x \in \mathbb{R}^d$.

Solution. Write $c_d := \Gamma(\frac{d+1}{2})\pi^{-(d+1)/2}$ and $\rho := (t^2+|x|^2)^{1/2}$. Then

$$P_t(x) = \frac{c_d\, t}{(t^2+|x|^2)^{(d+1)/2}} =: K(x,t), \tag{8}$$

and K extends to a C^∞ function on $\mathbb{R}^{d+1}\setminus\{0\}$ which is homogeneous of degree $-d$: $K(\lambda x, \lambda t) = \lambda^{-d}K(x,t)$ for $\lambda > 0$.

a) *Step 1: K is harmonic on* $\mathbb{R}^{d+1}\setminus\{0\}$. On $\mathbb{R}^n\setminus\{0\}$, $\Delta\rho^\alpha = \alpha(\alpha+n-2)\rho^{\alpha-2}$, so with $n = d+1$ the function $\Phi := \rho^{1-d}$ ($d \geq 2$), resp. $\Phi := \log\rho$ ($d = 1$), is harmonic, and

$$\partial_t\Phi = (1-d)\,t\,\rho^{-(d+1)} \ (d \geq 2), \qquad \partial_t\Phi = t\rho^{-2} \ (d = 1),$$

i.e. K is a constant multiple of $\partial_t\Phi$. As ∂_t commutes with $\Delta_{(x,t)}$ on the smooth function Φ, $\Delta_{(x,t)}K = 0$ on $\mathbb{R}^{d+1}\setminus\{0\}$.

Step 2: derivative bounds. Differentiating the homogeneity relation, each $\partial^\gamma K$ with $|\gamma| = k$ is homogeneous of degree $-d-k$ and continuous on the unit sphere, hence

$$|\partial^\gamma K(x,t)| \leq C_\gamma\, \rho^{-d-k} \qquad ((x,t) \neq 0). \tag{9}$$

Fix $(x_0,t_0) \in \mathbb{R}^{d+1}_+$ and the box $B := \{(x,t) : |x-x_0| \leq t_0/2,\ |t-t_0| \leq t_0/2\}$. For $(x,t) \in B$ and $y \in \mathbb{R}^d$, $t^2 + |x-y|^2 \geq \frac{1}{8}(t_0^2 + |x_0-y|^2)$ (if $|x_0-y| \geq t_0$ then $|x-y| \geq \frac{1}{2}|x_0-y|$; otherwise $t^2 \geq t_0^2/4$ and $|x_0-y|^2 < t_0^2$). By (9), for $|\gamma| = k \leq 2$,

$$\sup_{(x,t)\in B} |\partial^\gamma K(x-y,t)| \leq G_\gamma(y) := C_\gamma 8^{(d+k)/2}\big(t_0^2 + |x_0-y|^2\big)^{-(d+k)/2}. \tag{10}$$

G_γ is bounded and $\lesssim |x_0-y|^{-(d+k)}$ for $|x_0-y| \geq 1$, so $G_\gamma \in L^q(\mathbb{R}^d)$ for every $q \in [1,\infty]$. With q the conjugate exponent of p, Hölder gives

$$\int_{\mathbb{R}^d} G_\gamma(y)|f(y)|\,dy \leq \|G_\gamma\|_{L^q}\,\|f\|_{L^p} < \infty. \tag{11}$$

In particular ($\gamma = 0$) $u(x,t) = \int_{\mathbb{R}^d} K(x-y,t)f(y)\,dy$ converges absolutely for every $(x,t) \in \mathbb{R}^{d+1}_+$.

Step 3: differentiate under the integral. Let ∂ be one of $\partial_t, \partial_1, \ldots, \partial_d$, e the corresponding unit vector. For $0 < |h| < t_0/2$, by the mean value theorem there is $\theta = \theta(y,h) \in (0,1)$ with $(x_0,t_0) + \theta he \in B$ and

$$\frac{K\big((x_0,t_0)+he-(y,0)\big) - K\big((x_0,t_0)-(y,0)\big)}{h} = \partial K\big((x_0,t_0)+\theta he-(y,0)\big).$$

Hence

$$\Big|\frac{u((x_0,t_0)+he)-u(x_0,t_0)}{h}-\int_{\mathbb{R}^d}\partial K(x_0-y,t_0)f(y)\,dy\Big|$$
$$\leq \int_{\mathbb{R}^d}\big|\partial K\big((x_0,t_0)+\theta he-(y,0)\big)-\partial K(x_0-y,t_0)\big|\,|f(y)|\,dy,$$

where the integrand $\to 0$ as $h\to 0$ for every y (continuity of ∂K) and is $\leq 2G_\gamma(y)|f(y)|$, $|\gamma|=1$, integrable by (11). Dominated convergence along any $h_n\to 0$ gives $\partial u(x_0,t_0)=\int_{\mathbb{R}^d}\partial K(x_0-y,t_0)f(y)\,dy$. Repeating with kernel ∂K and $|\gamma|=2$ in (10), (11) gives the same for every second derivative $\partial^2\in\{\partial_t^2,\partial_1^2,\dots,\partial_d^2\}$.

Step 4. Summing and using Step 1,

$$\partial_t^2 u(x_0,t_0)+\sum_{j=1}^d \partial_j^2 u(x_0,t_0)=\int_{\mathbb{R}^d}\big(\Delta_{(x,t)}K\big)(x_0-y,t_0)\,f(y)\,dy=0.$$

b) This is Exercise 2b) with $k=P$: $P(x)=g(|x|)$ for $g(r)=c_d(1+r^2)^{-(d+1)/2}$, which is continuous and decreasing on $[0,\infty)$; $P\in L^1(\mathbb{R}^d)$ since $P(x)\leq c_d|x|^{-(d+1)}$ and $d+1>d$; $\int_{\mathbb{R}^d}P(x)\,dx=1$ by the choice of c_d (Lecture 3, Example 1); and $|P(x)|\leq g(|x|)$ trivially. Hence, for $p\in[1,\infty)$ and $f\in L^p(\mathbb{R}^d)$,

$$\lim_{t\to 0^+}u(x,t)=\lim_{t\to 0^+}(P_t*f)(x)=f(x)\qquad\text{for a.e. } x\in\mathbb{R}^d.$$

Exercise (4). a) Show that the functional p.v.$(1/x)$ given by

$$\Big\langle \text{p.v.}\tfrac{1}{x},\phi\Big\rangle=\lim_{\varepsilon\to 0^+}\int_{|x|>\varepsilon}\frac{\phi(x)}{x}\,dx\qquad(\phi\in\mathscr{S}(\mathbb{R}))$$

is well-defined (i.e. the limit above exists), and a bounded linear functional on $\mathscr{S}(\mathbb{R})$, hence a tempered distribution.

b) Show that $\widehat{\text{p.v.}(1/x)}=-i\pi\,\text{sgn}$, i.e.

$$\Big\langle \widehat{\text{p.v.}\tfrac{1}{x}},\phi\Big\rangle=\langle -i\pi\,\text{sgn},\phi\rangle\qquad(\phi\in\mathscr{S}(\mathbb{R})),\qquad \text{sgn}(\xi)=\begin{cases}1 & \xi>0\\ 0 & \xi=0\\ -1 & \xi<0.\end{cases}$$

Solution. **a)** Fix $\phi\in\mathscr{S}(\mathbb{R})$, $0<\varepsilon<1$. On $\{|x|>1\}$, $\big|\frac{\phi(x)}{x}\big|=\frac{|x\phi(x)|}{x^2}\leq\frac{p_{1,0}(\phi)}{x^2}$, integrable; on $\{\varepsilon<|x|<1\}$ the integrand is bounded by $p_{0,0}(\phi)/\varepsilon$. So

$$\int_{|x|>\varepsilon}\frac{\phi(x)}{x}\,dx=\int_{\varepsilon<|x|<1}\frac{\phi(x)}{x}\,dx+\int_{|x|>1}\frac{\phi(x)}{x}\,dx,\qquad \int_{|x|>1}\Big|\frac{\phi(x)}{x}\Big|\,dx\leq 2\,p_{1,0}(\phi).\tag{12}$$

Since $\int_{\varepsilon<|x|<1}\frac{dx}{x}=\log\frac{1}{\varepsilon}+\log\varepsilon=0$ ($\frac{1}{x}$ is odd), subtracting $\phi(0)\int_{\varepsilon<|x|<1}\frac{dx}{x}$,

$$\int_{\varepsilon<|x|<1}\frac{\phi(x)}{x}\,dx=\int_{\varepsilon<|x|<1}\psi(x)\,dx,\qquad \psi(x):=\frac{\phi(x)-\phi(0)}{x}=\frac{1}{x}\int_0^x\phi'(s)\,ds,\tag{13}$$

so $|\psi(x)|\leq p_{0,1}(\phi)$ for $x\neq 0$. Thus ψ is bounded on $\{0<|x|<1\}$, and $\big|\int_{0<|x|<1}\psi-\int_{\varepsilon<|x|<1}\psi\big|\leq\int_{0<|x|\leq\varepsilon}|\psi|\leq 2\varepsilon\,p_{0,1}(\phi)\to 0$ as $\varepsilon\to 0^+$. With (12), (13) the limit exists and

$$\Big\langle \text{p.v.}\tfrac{1}{x},\phi\Big\rangle=\int_{|x|<1}\frac{\phi(x)-\phi(0)}{x}\,dx+\int_{|x|>1}\frac{\phi(x)}{x}\,dx,\tag{14}$$

which is linear in ϕ, and by (13), (12),

$$\left|\left\langle \mathrm{p.v.}\tfrac{1}{x}, \phi \right\rangle\right| \le 2\, p_{0,1}(\phi) + 2\, p_{1,0}(\phi) \qquad (\phi \in \mathscr{S}(\mathbb{R})). \tag{15}$$

By (L4), (15) is the boundedness criterion with $C = 2$, $k = \ell = 1$; hence $\mathrm{p.v.}\frac{1}{x} \in \mathscr{S}'(\mathbb{R})$.

b) Fix $\phi \in \mathscr{S}(\mathbb{R})$. By (L4) and a), since $\widehat{\phi} \in \mathscr{S}(\mathbb{R})$ and $\widehat{\phi}/x$ is integrable on $\{|x| > \varepsilon\}$,

$$\left\langle \widehat{\mathrm{p.v.}\tfrac{1}{x}}, \phi \right\rangle = \lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} \frac{\widehat{\phi}(x)}{x}\, dx = \lim_{\varepsilon \to 0^+} \lim_{R \to \infty} \int_{\varepsilon<|x|<R} \frac{\widehat{\phi}(x)}{x}\, dx. \tag{16}$$

For $0 < \varepsilon < R < \infty$, $(x, \xi) \mapsto \phi(\xi) e^{-2\pi i x \xi}/x$ is integrable on $\{\varepsilon < |x| < R\} \times \mathbb{R}$, so by Fubini

$$\int_{\varepsilon<|x|<R} \frac{\widehat{\phi}(x)}{x}\, dx = \int_{\mathbb{R}} \phi(\xi)\, I_{\varepsilon,R}(\xi)\, d\xi, \qquad I_{\varepsilon,R}(\xi) := \int_{\varepsilon<|x|<R} \frac{e^{-2\pi i x \xi}}{x}\, dx. \tag{17}$$

Pairing x with $-x$ and substituting $s = 2\pi|\xi|x$, for $\xi \neq 0$

$$I_{\varepsilon,R}(\xi) = -2i \int_{\varepsilon}^{R} \frac{\sin(2\pi x \xi)}{x}\, dx = -2i\ \mathrm{sgn}(\xi)\big(S(2\pi R|\xi|) - S(2\pi\varepsilon|\xi|)\big), \qquad |I_{\varepsilon,R}(\xi)| \le 4, \tag{18}$$

and $I_{\varepsilon,R}(0) = 0$. Dominated convergence in (17) (dominating function $4|\phi| \in L^1$), first $R \to \infty$ then $\varepsilon \to 0^+$, using $S(t) \to \frac{\pi}{2}$ and $S(0) = 0$, gives

$$\left\langle \widehat{\mathrm{p.v.}\tfrac{1}{x}}, \phi \right\rangle = \int_{\mathbb{R}} \phi(\xi) \Big(-2i\ \mathrm{sgn}(\xi)\, \frac{\pi}{2}\Big) d\xi = -i\pi \int_{\mathbb{R}} \mathrm{sgn}(\xi)\, \phi(\xi)\, d\xi = \big\langle -i\pi\, \mathrm{sgn}, \phi \big\rangle.$$